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The Term Structure of Inflation Expectations

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Abstract

We present estimates of the term structure of inflation expectations, derived from an affine model of real and nominal yield curves. The model features stochastic covariation of inflation with the real pricing kernel. We fit the model not only to yields, but also to the yields' variance-covariance matrix, thus increasing identification power. We find that model-implied inflation expectations can differ substantially from breakeven inflation rates when market volatility is high. The model is estimated at a weekly frequency for use in real-time monetary policy analysis.

Key words: term structure of interest rates, inflation expectations, stochastic volatility, monetary policy

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1 Introduction

Central banks monitor financial market developments and economic data releases at relatively high frequencies. In the U.S., for example, Federal Reserve economists brief monetary policy makers at a weekly frequency, and the Federal Reserve's trading desk monitors market developments in real time. Understanding the linkages between asset price movements and macroeconomic developments is one of the key objectives of these monitoring efforts.

The evolution of breakeven inflation rates—the difference between nominal and real yields at different maturities—is an especially important indicator for the conduct of monetary policy, as breakeven inflation rates can be interpreted as measures of inflation expectations. Since the seminal work of Kydland and Prescott (1977) and Barro (1983), monetary economists have emphasized the importance of longer term inflation expectations, and recent literature suggests that the containment of long term inflation expectations is the most important objective in conducting monetary policy (see Woodford, 2003 and Bernanke et al., 2001 for summaries).

Breakeven inflation rates measure inflation expectations with a number of biases, some of which have been previously documented (see Barr and Campbell, 1997, Elsasser and Sack, 2004, and Gurkaynak, Sack, and Wright, 2007). First, inflation linked benchmark securities are typically less liquid than nominal on-the run Treasuries. Second, the coupons of nominal and real securities with similar maturities are often different, leading to differences in duration.

The focus of the current paper is the difference between expected inflation and breakeven inflation due to inflation risk. We analyze the term structure of breakeven inflation computed as the difference between a zero coupon, off-the-run nominal Treasury yield curve and a zero coupon, real TIPS curve.[1] This breakeven term structure is sometimes called the term structure of implied inflation. Implied inflation is a better measure of inflation expectations than breakeven inflation, as it adjusts for liquidity differences and for differences in durations.[2] However, breakeven inflation rates of zero-coupon off-the-run curves (i.e., implied inflation) still are not pure measures of inflation expectations. This is because the absence of arbitrage implies that there is a wedge between breakeven inflation and expected inflation which is the inflation risk premium:

$$\text{Expected Inflation} = \text{Breakeven Inflation - Inflation Risk Premium}$$

In this paper, we develop an affine term structure model that captures the dynamics of real and nominal yields curves, as well as the evolution of their variance-covariance matrix. This is important, as the inflation risk premium is proportional to the conditional covariance of the real pricing kernel and inflation. In order to increase the power for identifying the inflation risk premium, we match both the term structure of the yield curves, and the term structure of variances and covariances.

We find an inflation risk premium that has varied substantially in times of high market volatility, but has otherwise been relatively stable. The order of magnitude of the inflation risk premium is comparable to other recent estimates in studies that use inflation protected bonds (D'Amico, Kim, and Wei, 2008 and Christensen, Lopez, and Rudebusch, 2008), but it is smaller than estimates that use nominal bonds and inflation over longer time periods (see Buraschi and Jiltsov, 2005, and Ang, Bekaert, and Wei, 2006, Campbell, Sundarem, and Viceira, 2007).

[1] The nominal term structure is from Gurkaynak, Sack, and Wright (2006), the real term structure is from Gurkaynak, Sack, and Wright (2007).

[2] In the remainder of the paper, we use the terminology "implied inflation" and "breakeven inflation" interchangably, as our yield curves are off-the-run zero coupon curves.

We use only real and nominal yields to estimate the term structure of inflation expectations. Our requirement to match the variance-covariance matrix of the real and nominal yield curves provides us with enough identification power to estimate expected inflation. In comparison, some recent work incorporates estimates of inflation expectations from survey data to achieve identification (see Joyce, Lildholdt, and Sorensen, 2007, Hördahl and Tristani, 2007, D'Amico, Kim, and Wei, 2008, and Chernov and Mueller, 2008). We view those papers as complementary identification strategies.

We fit our model to an estimated variance-covariance matrix of the real and nominal yield curves. Alternatively, options data could be used to obtain information about second moments. For example, Goldstein and Collin-Dufresne (2002) use interest rate caps and floors to fit an affine term structure of the nominal yield curve that features stochastic volatility. The advantage of our approach is that our data is readily available. To obtain an implied covariance between nominal and real yields, one would have to have a time series of options on inflation swaps, which have only been liquid, for a short period of time.

The remainder of the paper is organized in five sections. In Section 2, we discuss the nominal and real yield curves and derive the relationship between breakeven inflation and expected inflation from no arbitrage. We present our model in Section 3, and our estimation results in Section 4. In section 5, we provide robustness checks, and Section 6 concludes.

2 Breakevens and the Inflation Risk Premium

2.1 The nominal and real yield curves

Breakeven inflation provides a measure of inflation expectations that can be tracked at high frequencies. However, using breakevens to measure inflation expectations is problematic. On-the-run Treasuries are more liquid than benchmark TIPS. On average, the difference between the on-the-run and the off-the-run yield is 6 basis points since the beginning of 2003, with a daily standard deviation of 2.2% (see also Fleming, 2003, and Krishnamurthy, 2002, for analysis of the on-the-run/off-the-run Treasury yield spread). In periods of financial market turbulence, the on-the-run/off-the-run spread tends to widen. However, financial market turbulences are times when inflation expectations potentially change, and when variances and covariances and hence risk premia tend to change. An increase in the on-the-run/off-the-run spread during those times tends to understate inflation expectations relative to breakeven inflation computed from on-the-run bonds. In addition, coupons of nominal Treasuries and TIPS with similar maturities are often different. A ten-year breakeven spread typically has a duration that is shorter than ten years, and the difference in duration of the nominal and the real yield introduces a bias. The average wedge between par and zero coupon yields with a ten year maturity is 44 basis points since the beginning of 2004.

In this paper, we use the zero-coupon, off-the run term structure of nominal Treasury yields computed by Gurkaynak, Sack, and Wright (2006), and the zero coupon real term structure computed from TIPS real yields by Gurkaynak, Sack, and Wright (2007). The yield data is daily and spans from January 11, 2003 to December 31, 2009. We estimate the model at a weekly frequency, using the last day of each week to construct our dataset, providing us with 365 weeks. We plot the yields of TIPS and

Figure 1: 10-Year Treasury Yield, TIPS Yield, and Breakeven.

Treasury securities together with the breakevens for the 10 year maturity in Figure 1, and provide summary statistics in Table 1.

2.2 The inflation risk premium

We denote the maturity of a bond by τ. The price of a nominal bond that pays \$1 at time $t+\tau$ is P_t^τ. The price of a real bond that pays \$1 at time $t+\tau$ is R_t^τ. The Daily Reference Level of the CPI is denoted by Q_t. The absence of arbitrage implies that there exists a real discount factor M_t such that (see, for example Dybvig and Ross, 1987, for an exposition of the Fundamental Theorem of Arbitrage Pricing):

$$P_t^\tau = E_t\left[\frac{M_{t+\tau}}{M_t}\frac{Q_t}{Q_{t+\tau}}\right], \qquad R_t^\tau = E_t\left[\frac{M_{t+\tau}}{M_t}\right] \tag{1}$$

Table 1: **Summary Statistics for Yields.** The table reports summary statistics for nominal, zero coupon, off-the-run Treasury yields and real, zero coupon TIPS yields for maturities 3, 4, 5, 6, 7, 8, 9, and 10 years. The data is weekly from 1/11/2003-12/31/2009. Source: Board of Governors of the Federal Reserve. The nominal term structure is from Gurkaynak, Sack, and Wright (2006), the real term structure is from Gurkaynak, Sack, and Wright (2007).

	Nominal Yields					Real Yields				
Maturity	Mean	Median	Std.	Max	Min	Mean	Median	Std.	Max	Min
3	3.06	2.93	1.23	5.17	0.79	1.27	1.02	0.93	4.24	-0.47
4	3.29	3.23	1.05	5.13	1.10	1.43	1.21	0.76	3.84	-0.16
5	3.51	3.48	0.89	5.11	1.43	1.58	1.42	0.64	3.68	0.12
6	3.72	3.72	0.76	5.12	1.75	1.70	1.59	0.56	3.67	0.38
7	3.91	3.92	0.65	5.14	2.05	1.81	1.75	0.49	3.67	0.61
8	4.09	4.10	0.57	5.18	2.31	1.90	1.87	0.45	3.66	0.81
9	4.24	4.27	0.50	5.22	2.53	1.98	1.96	0.41	3.63	0.99
10	4.38	4.42	0.46	5.26	2.72	2.04	2.02	0.37	3.58	1.14

We denote the continuously compounded yield of the nominal bond $y_t^\tau = -(1/\tau)\ln(P_t^\tau)$, and of the inflation linked bond by $r_t^\tau = -(1/\tau)\ln(R_t^\tau)$. We further denote the logarithmic inflation rate by π_t, and the rate of change of (the negative of) the pricing kernel by m_t:

$$m_{t+1} = -(\ln M_{t+1} - \ln M_t) \text{ and } \pi_{t+1} = \ln Q_{t+1} - \ln Q_t \tag{2}$$

Using these expressions (2) in equation (1) gives:

$$y_t^\tau = -\frac{1}{\tau}\ln E_t\left[\exp\left(-\Sigma_{s=1}^\tau m_{t+s} - \Sigma_{s=1}^\tau \pi_{t+s}\right)\right] \tag{3a}$$

$$r_t^\tau = -\frac{1}{\tau}\ln E_t\left[\exp\left(-\Sigma_{s=1}^\tau m_{t+s}\right)\right] \tag{3b}$$

We will define the inflation risk premium at maturity τ as the difference between breakeven inflation and expected inflation generated from the model:

$$\underbrace{IRP_t^\tau}_{\text{Inflation Risk Premium}} = \underbrace{y_t^\tau - r_t^\tau}_{\text{Breakeven Inflation}} - \underbrace{\frac{1}{\tau} E_t\left[\Sigma_{s=1}^{\tau} \pi_{t+s}\right]}_{\text{Expected Inflation}} \tag{4}$$

In order to gain insight into the drivers of the inflation risk premium, it is useful to derive the closed form expression in the case of a one period bond. To do so, we follow the affine term structure literature (see Piazzesi, 2003, and Singleton, 2006 for surveys) and assume that shocks to inflation and the real pricing kernel are conditionally normal. For a one period bond, the inflation risk premium IRP_t^1 is then equal to the covariance between the real pricing kernel and inflation, minus a convexity adjustment:

$$IRP_t^1 = Cov_t\left(\pi_{t+1}, -m_{t+1}\right) - \frac{1}{2} Var_t\left(\pi_{t+1}\right) \tag{5}$$

The inflation risk premium compensates investors for the risk that inflation varies in the future. The inflation risk premium is positive if inflation covaries negatively with the pricing kernel M. In a consumption based asset pricing framework, the pricing kernel is related to the growth rate of the marginal utility of consumption. Inflation tends to be high when consumption growth is high, and the marginal utility of consumption is low. So the negative of the pricing kernel tends to covary positively with inflation, and we would expect the inflation risk premium to be positive. The inflation risk premium also contains a convexity adjustment is proportional to the conditional variance of inflation.

For multiperiod bonds, there is no closed form expression for the inflation risk premium IRP_t^τ. Instead, in the remainder of the paper, we will compute the inflation risk premium IRP_t^τ as the difference between breakeven inflation at maturity τ, and the expected inflation estimate $\frac{1}{\tau}E_t\left[\Sigma_{s=1}^{\tau}\pi_{t+s}\right]$ implied by the estimated model.

2.3 The term structure of yield second moments

Equation (5) suggests that the inflation risk premium is proportional to the covariation between future inflation and the future real pricing kernel. In order to identify this covariation precisely, we match the variance-covariance matrix of nominal and real yields. Second moments cannot be observed directly, but they can be estimated precisely when the frequency of observed yields is high (see Merton, 1980). We use the multivariate GARCH model proposed by Engle and Kroner (1995) to estimate the dynamics of the variance-covariance matrix for nominal and real yields, for each maturity.

$$\left[\begin{array}{cc}\left(\hat{\sigma}_{t+1}^{y}\right)^2 & \hat{\sigma}_{t+1}^{yr}\\ \hat{\sigma}_{t+1}^{yr} & \left(\hat{\sigma}_{t+1}^{r}\right)^2\end{array}\right]=A_0'A_0+A_1'\left[\begin{array}{cc}\left(\hat{\sigma}_{t}^{y}\right)^2 & \hat{\sigma}_{t}^{yr}\\ \hat{\sigma}_{t}^{yr} & \left(\hat{\sigma}_{t}^{r}\right)^2\end{array}\right]A_1+A_2'\left[\begin{array}{cc}\left(\hat{\varepsilon}_{t+1}^{y}\right)^2 & \hat{\varepsilon}_{t+1}^{y}\hat{\varepsilon}_{t+1}^{r}\\ \hat{\varepsilon}_{t+1}^{y}\hat{\varepsilon}_{t+1}^{r} & \left(\hat{\varepsilon}_{t+1}^{r}\right)^2\end{array}\right]A_2$$

where $\hat{\varepsilon}_{t+1}^{y}$ is the residual of a regression of nominal yields on lagged nominal and real yields (maturity by maturity), and $\hat{\varepsilon}_{t+1}^{r}$ are the residuals of a regression of real yields on lagged real and nominal yields (again maturity by maturity). Furthermore, $\left(\hat{\sigma}_{t}^{y}\right)^2 = Var_t\left(y_{t+1}\right)$, $\hat{\sigma}_{t}^{yr} = Cov_t\left(y_{t+1},\ r_{t+1}\right)$, $\left(\hat{\sigma}_{t}^{r}\right)^2 = Var_t\left(r_{t+1}\right)$. Because the evolution of the variance-covariance matrix is according to a quadratic form, it is assured to be positive definite. We provide summary statistics of the estimated variances and covariances in Table 2.3 and a plot of the conditional variances and covariances for the ten year maturity in Figure 2.

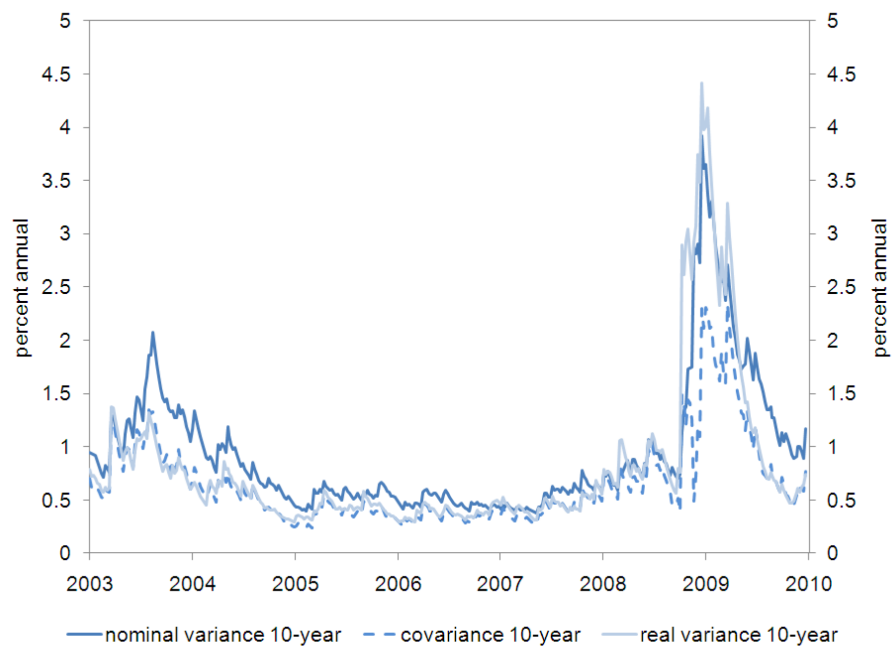

Figure 2: Estimated Conditional Variances and Covariances for the 10-Year Nominal and Real Yields.

Table 2: Summary Statistics for Variances and Covariances of Yields.

	Nominal Yield Variance ($\times 10^2$)					Real Yield Variance ($\times 10^2$)				
Maturity	Mean	Median	Std.	Max	Min	Mean	Median	Std.	Max	Min
3	1.78	1.67	0.65	4.22	0.92	2.85	1.86	3.09	22.28	0.87
4	1.85	1.85	0.69	4.14	0.88	2.36	1.69	2.14	12.79	0.76
5	1.86	1.82	0.75	3.86	0.84	2.14	1.59	1.88	10.39	0.70
6	1.85	1.73	0.83	4.04	0.77	1.99	1.47	1.79	10.21	0.64
7	1.84	1.64	0.90	4.81	0.76	1.86	1.35	1.69	9.97	0.62
8	1.83	1.53	1.03	5.83	0.73	1.75	1.26	1.61	9.69	0.61
9	1.82	1.44	1.12	6.69	0.73	1.64	1.15	1.54	9.35	0.55
10	1.81	1.39	1.20	7.55	0.73	1.54	1.09	1.40	8.49	0.56

	Yield Covariance ($\times 10^2$)				
Maturity	Mean	Median	Std.	Max	Min
3	1.28	1.14	0.62	3.93	0.24
4	1.33	1.19	0.60	3.62	0.44
5	1.36	1.24	0.62	3.41	0.35
6	1.36	1.21	0.67	3.69	0.35
7	1.34	1.19	0.70	4.06	0.33
8	1.32	1.10	0.76	4.42	0.38
9	1.29	1.07	0.79	4.58	0.42
10	1.26	1.01	0.78	4.52	0.45

3 Modeling Inflation Expectations

3.1 State variables and pricing kernel specification

We allow for two inflation factors π_t^1, π_t^2 and for two real factors m_t^1, m_t^2. We normalize the factors such that $\pi_t = \pi_t^1 + \pi_t^2$ and $m_t = m_t^1 + m_t^2$. We further model a factor σ_t^2 that guides the dynamic evolution of the variances and covariances of the state variables.[3]

[3] In earlier versions of this paper, we presented specifications with multiple factors in the variance-covariance matrix. The modeling of only one second moment factor makes the model more parsimonious, but does not reduce the fit of the model. We would like to thank an anonymous referee for suggesting to use only one second moment factor.

The vector of state variables is:

$$X_t = \left[\begin{array}{ccccc} \pi_t^1 & \pi_t^2 & m_t^1 & m_t^2 & \sigma_t^2 \end{array} \right]'$$

with dynamic evolution:

$$\begin{aligned} X_{t+1} &= \mu + \Phi X_t + \Sigma_t \epsilon_{t+1} \text{ where } \epsilon_{t+1} \sim N\left(0, I_5\right) &(6) \\ vec\left(\Sigma_t \Sigma_t'\right) &= S_0 + S_1 X_t &(7) \end{aligned}$$

To estimate the model (6), we impose a number of restrictions on μ, Φ, S_0 and S_1.

Following Duffie and Kan (1996) and Duffee (2002), we model the pricing kernel allowing for flexible prices of risk. In particular, we specify the rate of change of (the negative of) the log pricing kernel m_t from equation (2) as:

$$m_{t+1} = r_t^1 + \frac{1}{2}\lambda_t' \Sigma_t^{-1} \Sigma_t'^{-1} \lambda_t + \lambda_t' \Sigma_t^{-1} \epsilon_{t+1} \tag{8}$$

where λ_t is the time-varying market price of risk, and r_t is the one-period real short rate, both of which are affine functions of state variables:

$$\lambda_t = \lambda_0 + \lambda_1 X_t, \qquad r_t^1 = \delta_0 + \delta_1 X_t \tag{9}$$

3.2 No-arbitrage pricing restrictions

Because the evolution of state variables is conditionally Gaussian, and prices of risk are conditionally Gaussian, nominal and real yields y_t^τ and r_t^τ are affine functions of

X_t. To show this, we start by writing $(3a)$ recursively:

$$y_t^\tau = -\frac{1}{\tau}\ln E_t\left[\exp -m_{t+1} - \pi_{t+1} - (\tau-1)\, y_{t+1}^{\tau-1}\right] \tag{10}$$

$$r_t^\tau = -\frac{1}{\tau}\ln E_t\left[\exp -m_{t+1} - (\tau-1)\, r_{t+1}^{\tau-1}\right] \tag{11}$$

The affine pricing function then takes the following form:

$$\begin{aligned} y_t^\tau &= \tfrac{1}{\tau}C_{0y}^\tau + \tfrac{1}{\tau}C_{1y}^\tau X_t \\ r_t^\tau &= \tfrac{1}{\tau}C_{0r}^\tau + \tfrac{1}{\tau}C_{1r}^\tau X_t \end{aligned} \tag{12}$$

For real yields, we then find:

$$C_{0r}^\tau + C_{1r}^\tau X_t = -\ln E_t\left[\exp -r_t^1 - \frac{1}{2}\lambda_t'\Sigma_t^{-1}\Sigma_t'^{-1}\lambda_t - \lambda_t'\Sigma_t^{-1}\epsilon_{t+1} - C_{0r}^{\tau-1} - C_{1r}^{\tau-1}X_{t+1}\right] \tag{13}$$

Using the properties of the moment generating function of the normal distribution and collecting terms gives:

$$\begin{aligned} C_{0r}^\tau + C_{1r}^\tau X_t &= \delta_0 + C_{0r}^{\tau-1} + C_{1r}^{\tau-1}(\mu-\lambda_0) - \frac{1}{2}C_{1r}^{\tau-1}\otimes C_{1r}^{\tau-1}S_0 \\ &\quad + \left[C_{1r}^{\tau-1}(\Phi-\lambda_1) + \delta_1 - \frac{1}{2}C_{1r}^{\tau-1}\otimes C_{1r}^{\tau-1}S_1\right]X_t \end{aligned}$$

Matching terms gives:

$$C_{0r}^\tau = C_{0r}^{\tau-1} + C_{1r}^{\tau-1}(\mu-\lambda_0) - \frac{1}{2}C_{1r}^{\tau-1}\otimes C_{1r}^{\tau-1}S_0 + \delta_0 \tag{14}$$

$$C_{1r}^\tau = C_{1r}^{\tau-1}(\Phi-\lambda_1) - \frac{1}{2}C_{1r}^{\tau-1}\otimes C_{1r}^{\tau-1}S_1 + \delta_1 \tag{15}$$

For nominal yields, the no arbitrage recursion is:

$$\tau y_t^\tau = -\frac{1}{\tau} \ln E_t \left[\exp\left(-\Sigma_{s=1}^{\tau} m_{t+s} - \Sigma_{s=1}^{\tau} \pi_{t+s}\right)\right] \tag{16}$$

Replacing the guess for the yield from (12) into (16) gives:

$$C_{0y}^\tau + C_{1y}^\tau X_t = -\ln E_t \left[\exp - m_{t+1} - \phi X_{t+1} - C_{0y}^{\tau-1} - C_{1y}^{\tau-1} X_{t+1}\right]$$

where $\phi = [1\ 0\ 0\ ...]$. Then we find:

$$\begin{aligned} C_{0y}^\tau + C_{1y}^\tau X_t &= \delta_0 + C_{0y}^{\tau-1} + \left(C_{1y}^{\tau-1} + \phi\right)(\mu - \lambda_0) - \frac{1}{2}\left(C_{1y}^{\tau-1} + \phi\right) \otimes \left(C_{1y}^{\tau-1} + \phi\right) S_0 \\ &\quad + \left[\left(C_{1y}^{\tau-1} + \phi\right)(\Phi - \lambda_1) - \frac{1}{2}\left(C_{1y}^{\tau-1} + \phi\right) \otimes \left(C_{1y}^{\tau-1} + \phi\right) S_1 + \delta_1\right] X_t \end{aligned}$$

Matching coefficients, and we find:

$$C_{0y}^\tau = C_{0y}^{\tau-1} + \left(C_{1y}^{\tau-1} + \phi\right)(\mu - \lambda_0) - \frac{1}{2}\left(C_{1y}^{\tau-1} + \phi\right) \otimes \left(C_{1y}^{\tau-1} + \phi\right) S_0 + \delta_0 \tag{17}$$

$$C_{1y}^\tau = \left(C_{1y}^{\tau-1} + \phi\right)(\Phi - \lambda_1) - \frac{1}{2}\left(C_{1y}^{\tau-1} + \phi\right) \otimes \left(C_{1y}^{\tau-1} + \phi\right) S_1 + \delta_1 \tag{18}$$

For notational convenience we stack $C_0^\tau = [\ C_{0y}^\tau \quad C_{0r}^\tau\ ]'$ and $C_1^\tau = [\ C_{1y}^\tau \quad C_{1r}^\tau\ ]'$ so that

$$\begin{pmatrix} y_t^\tau \\ r_t^\tau \end{pmatrix} = \frac{1}{\tau} C_0^\tau + \frac{1}{\tau} C_1^\tau X_t \tag{19}$$

It directly follows from (19) that the conditional variance-covariance matrix of nominal and real yields is also affine:

$$vec\begin{bmatrix} Var_t\left(y_{t+1}^{\tau}\right) & Cov_t\left(y_{t+1}^{\tau}, r_{t+1}^{\tau}\right) \\ Cov_t\left(y_{t+1}^{\tau}, r_{t+1}^{\tau}\right) & Var_t\left(r_{t+1}^{\tau}\right) \end{bmatrix} = \frac{1}{\tau^2}\left(C_1^{\tau} \otimes C_1^{\tau}\right) S_0 + \frac{1}{\tau^2}\left(C_1^{\tau} \otimes C_1^{\tau}\right) S_1 X_t \tag{20}$$

The requirement for the term structure to be arbitrage free thus not only imposes consistency of pricing across the yield curve, but also consistency of the term structure of variance-covariance matrices across maturities.

3.3 Estimation method

We estimate the model via maximum likelihood and obtain the state variables from the Kalman filter. The state space representation of our model is:

$$Y_t = C_0 + C_1 X_t + \upsilon_t \tag{21a}$$

$$X_{t+1} = \mu + \Phi X_t + \Sigma_t \epsilon_{t+1} \tag{21b}$$

$$vec(\Sigma_{t+1}\Sigma_{t+1}') = S_0 + S_1 X_t \tag{21c}$$

where $Y_t = \begin{bmatrix} y_t & r_t & Var_{t-1}\left(y_t\right) & Cov_{t-1}\left(y_t,\ r_t\right) & Var_{t-1}\left(r_t\right) \end{bmatrix}'$ and C_0 and C_1 stack coefficients of (19) and (20) across maturities. We treat the variance-covariance matrix of nominal / real yield pairs as observable. We assume that the pricing error υ_t is normally distributed with constant, diagonal covariance matrix R. Based on state space representation in (21), we filter the factors according to the Kalman filter:

$$\hat{X}_t = \mu + \Phi \hat{X}_{t-1} + K_t\left(Y_t - C_0 - C_1\left(\mu + \Phi \hat{X}_{t-1}\right)\right) \tag{22}$$

where K_t is the Kalman gain (see Hamilton, 1994). Given estimates of the latent factors $\hat{X}_t$, the parameters $\Theta = \{\mu, \Phi, S_0, \delta_0, \delta_1, \lambda_0, \lambda_1\}$ are estimated by maximum likelihood, based on the conditional distribution of $Y_t|Y_{t-1}$ for each observation. The conditional distribution of Y_t is $N\left(\hat{Y}_t|Y_{t-1}, \Omega_t^Y\right)$ with $\hat{Y}_t|Y_{t-1} = C_0 + C_1\hat{X}_{t-1}|Y_{t-1}$ and $\Omega_t^Y = C_1 Var\left(X_t|Y_{t-1}\right) C_1 + R$, and we assume that the variance-covariance matrix of the observation errors, R, is constant and diagonal. The log likelihood function is then:

$$L(\Theta) = -\sum_{t=1}^{T}\left(\log\left|\Omega_t^Y\right| + \left(Y_t - \hat{Y}_t|Y_{t-1}\right)\left(\Omega_t^Y\right)^{-1}\left(Y_t - \hat{Y}_t|Y_{t-1}\right)'\right) \tag{23}$$

4 Estimating Inflation Expectations

4.1 Pricing

Tables 3, 4, and 5 present the parameter estimates. We use annualized percent yields in the estimation, so the long-run mean of estimated inflation is 1.89% annual. We reject a unit root for the inflation process, but do find inflation to be persistent. The two factors of the real pricing kernel m^1 and m^2 correspond to the level and slope factors of the real term structure, while the two inflation factors π_1 and π_2 represent the level and slope factors of inflation.

We match yields and second moments well. This can be seen in Tables 6 and 7 that give the summary statistics of pricing errors. We find small errors for yields and the variance-covariance of yields.

In the existing literature, term structure models are usually fitted to match yields. In this paper, we match time varying variance-covariance matrix of yields as well. This procedure provides us with greater confidence about the accuracy of our estimated inflation risk premium. For the Treasury yield curve, some recent papers have used

Table 3: Parameter Estimates.

	Coefficient Estimate	Std. Err.
$\mu =$	$\begin{pmatrix} 0.0073 \\ -0.0069 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0.0026 \\ 0.0053 \\ . \\ . \\ . \end{pmatrix}$
$\Phi =$	$\begin{pmatrix} 0.9962 & . & . & . & . \\ -0.0031 & 0.9669 & . & . & . \\ -0.0004 & . & 0.9955 & . & . \\ 0.0082 & 0.0720 & 0.0211 & 0.9965 & . \\ -0.0024 & -0.0416 & . & -0.0018 & 0.9999 \end{pmatrix}$	$\begin{pmatrix} 0.0008 & . & . & . & . \\ 0.0023 & 0.1050 & . & . & . \\ 0.0004 & . & 0.0014 & . & . \\ 0.0058 & 0.0455 & 0.0203 & 0.0095 & . \\ 0.0025 & 0.0484 & . & 0.0027 & 0.0057 \end{pmatrix}$

options data in order to gauge information about second moments (see Goldstein and Collin-Dufresne, 2002 and Bikbov and Chernov, 2005).

4.2 Factors

The filtered factors of the pricing kernel are plotted in Figures 3 and 4. The real pricing kernel m is determining the real term structure. It declined in 2003 and 2004, and again in the second half of 2007, when real activity was slowing. The real factor m surged during the fall of 2008. In a simple consumption based asset pricing model, m is proportional to the growth rate of consumption. In more elaborate habit formation models, m is proportional to the deviation of consumption growth from a moving average of consumption growth (the habit). The latter model of the real kernel might be consistent with our estimated m, but we do not investigate this route further (see Wachter, 2006 for a term structure model with habit formation).

The filtered inflation factor π increases in 2003-2005, and declines slowly from

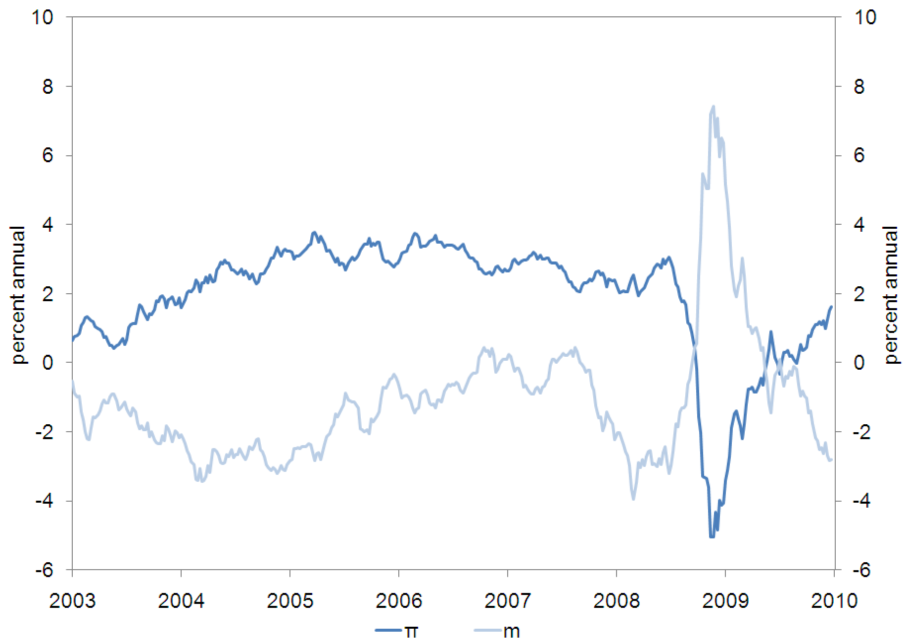

Figure 3: The Inflation and Real Factors.

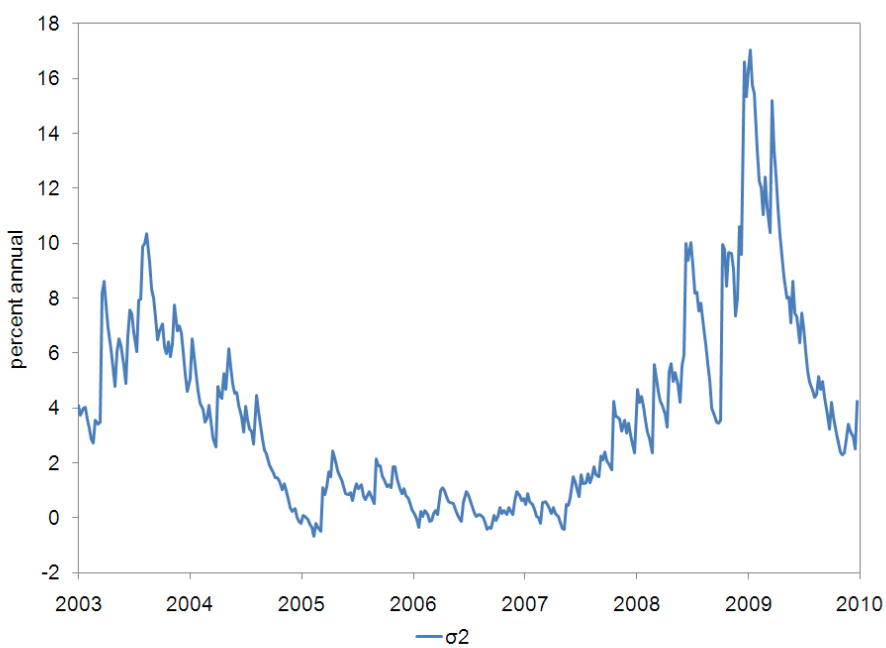

Figure 4: The Variance Factor.

Table 4: **Parameter Estimates (Continued).**

	Coefficient Estimate	Std. Err.
$\tilde{S}_0 =$	$\begin{pmatrix} -0.0609 & & & & \\ & 0 & & & \\ & & -0.0025 & & \\ & & & 0.0828 & \\ & & & & -0.0177 \end{pmatrix}$	$\begin{pmatrix} 0.0000 & . & . & . & . \\ . & . & . & . & . \\ . & . & 0.0204 & . & . \\ . & . & . & 0.0546 & . \\ . & . & . & . & 0.0198 \end{pmatrix}$
$\tilde{S}_1 =$	$\begin{pmatrix} 0 & -0.0969 & 0.0598 & -0.7793 \\ & -0.8353 & 0 & -0.2684 \\ & & -0.2308 & 0.1671 \\ & & & 0.9703 \end{pmatrix}$	$\begin{pmatrix} . & 0.4380 & 0.0588 & 0.4032 \\ . & 0.6562 & . & 0.3600 \\ . & . & 0.1220 & 0.1464 \\ . & . & . & 1.0158 \end{pmatrix}$
$\delta_0 =$	0	.
$\delta_1 =$	$\begin{pmatrix} 0 \\ -12.8330 \\ 0 \\ 0.5649 \\ 0 \end{pmatrix}$	$\begin{pmatrix} . \\ 2.0821 \\ . \\ 0.9156 \\ . \end{pmatrix}$

2005 to 2007. The inflation factor declines sharply in the fall of 2008, corresponding to a marked shift in inflation expectations. The variance factor σ^2 also shows a dramatic movement in 2008-2009, as can be seen in Figure 4. The volatility factor was particularly low in the 2005-2007 period, corresponding to the credit boom.

4.3 Expected inflation and forward inflation

Expected inflation can be computed from the dynamics of the state equation (6):

$$\frac{1}{\tau} E_t \left(\Sigma_{s=1}^{\tau} \pi_{t+s} \right) = [1\ 1\ 0\ 0...] \left(\tilde{C}_0^{\tau} + \tilde{C}_1^{\tau} X_t \right) \tag{24}$$

where $\tilde{C}_0^{\tau} = \tilde{C}_0^{\tau-1} + \left(I_5 + \tilde{C}_1^{\tau-1} \right) \mu$, $\tilde{C}_1^{\tau} = \left(I_5 + C_1^{\tau-1} \right) \Phi$, $\tilde{C}_0^0 = \tilde{C}_1^0 = 0$.

Table 5: Parameter Estimates (Continued).

	Coefficient Estimate	Std. Err.
$\lambda_0 =$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}'$	.
$\lambda_1 =$	$\begin{pmatrix} 0 & 0.0167 & 0 & 0 & 0 \\ & 0.0350 & 0.0041 & 0.0035 & -0.0025 \\ & & 0 & 0 & -0.0065 \\ & & & -0.0019 & 0.0066 \\ & & & & 0 \end{pmatrix}$	$\begin{pmatrix} . & 0.0169 & . & . & . \\ . & 0.1448 & 0.0034 & 0.0030 & 0.0088 \\ . & . & . & . & 0.0071 \\ . & . & . & 0.0093 & 0.0171 \\ . & . & . & . & . \end{pmatrix}$

Table 6: Pricing Errors.

	Nominal Yields			Real Yields		
Maturity	Mean	Median	Std.	Mean	Median	Std.
3	0.01	0.02	0.07	0.01	0.01	0.09
4	-0.01	-0.01	0.03	-0.01	-0.01	0.03
5	-0.01	-0.01	0.03	-0.01	-0.01	0.03
6	0.00	0.00	0.04	0.00	0.00	0.04
7	0.01	0.01	0.04	0.00	0.00	0.04
8	0.01	0.01	0.04	0.00	0.00	0.04
9	0.00	0.01	0.04	0.00	0.01	0.05
10	-0.01	-0.01	0.05	0.00	0.00	0.05

We plot the 10 year breakeven forward rates together with the expected inflation forward rates in Figure 5, and the 5-10 year forward breakeven and expected inflation forward rates in Figure 6.

The inflation risk premium is somewhat larger for the 5-10 year forward than for the 10-year maturity. In both cases, the inflation risk premium increased sharply during the financial crisis of the fall of 2008, corresponding to a period or large uncertainty. In general, there is a tight link between the inflation risk premium and the variance factor σ^2.

The adjustment to breakeven inflation (i.e. the difference between breakeven infla-

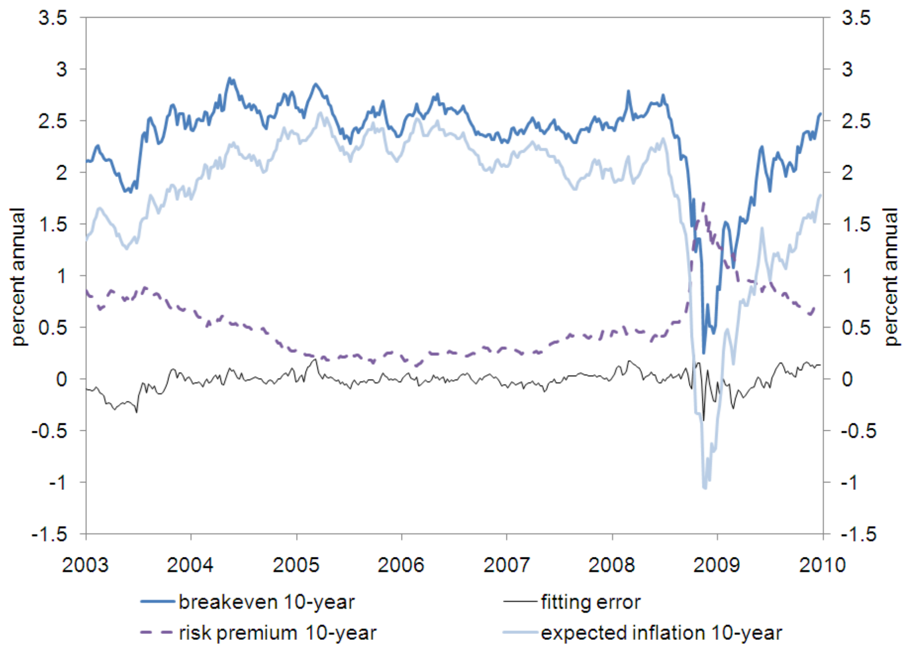

Figure 5: Expected Inflation, Inflation Risk Premium, and Breakeven Inflation for the 10-Year Maturity.

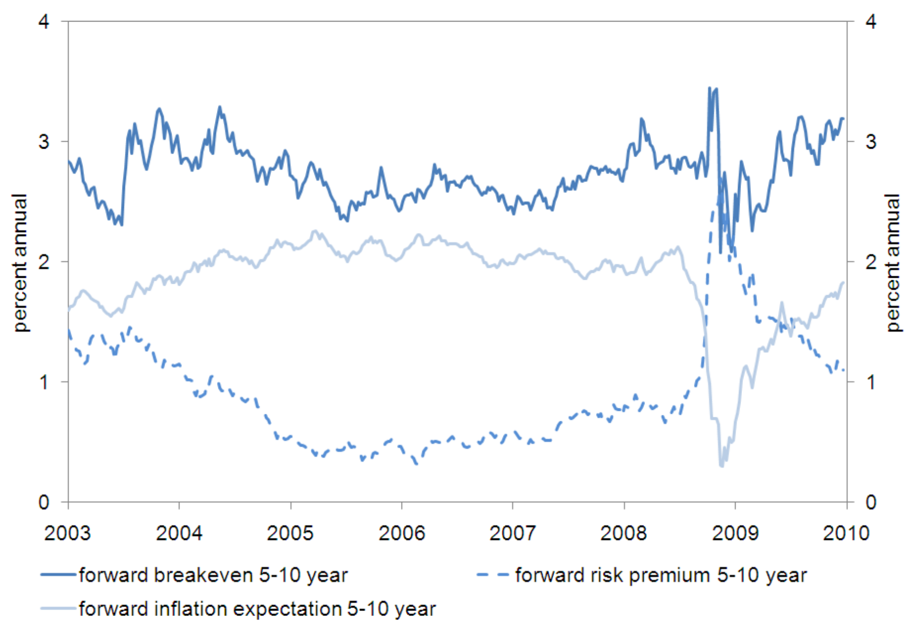

Figure 6: Expected Inflation, Inflation Risk Premium, and Breakeven Inflation 5-10 Year Forward.

Table 7: Pricing Errors (continued).

	Nominal Variance ($\times 10^4$)			Real Variance ($\times 10^4$)		
Maturity	Mean	Median	Std.	Mean	Median	Std.
3	-3.82	-15.60	44.43	0.57	2.52	228.72
4	1.34	-4.90	39.04	-10.21	-1.13	129.92
5	2.16	-1.57	32.56	-10.37	-6.23	99.23
6	1.64	-2.13	26.39	-7.25	-6.29	88.37
7	0.13	-0.55	25.64	-4.75	-4.79	83.34
8	-1.00	0.02	31.64	-2.50	-2.90	79.77
9	-1.15	1.45	39.58	-1.89	-3.08	77.01
10	1.08	3.43	48.29	-2.63	-1.20	69.69

	Covariance ($\times 10^4$)		
Maturity	Mean	Median	Std.
3	-0.19	-8.50	53.83
4	2.44	-0.79	43.73
5	0.93	0.74	33.05
6	-0.41	-0.23	23.16
7	-0.70	0.87	18.24
8	0.13	-0.17	18.35
9	0.00	0.89	20.28
10	0.26	3.31	22.24

tion and expected inflation) correlates highly with market based measures of implied volatility. This can be seen in Figure 7 where we plot the 5-10 year inflation risk premium together with the S&P 500 implied volatility (VIX) computed by the Chicago Board Options Exchange (CBOE). Note that the tight link between the estimated intflation risk premium and the VIX has not been imposed by the model in any way, but is rather an outcome of the model estimates.

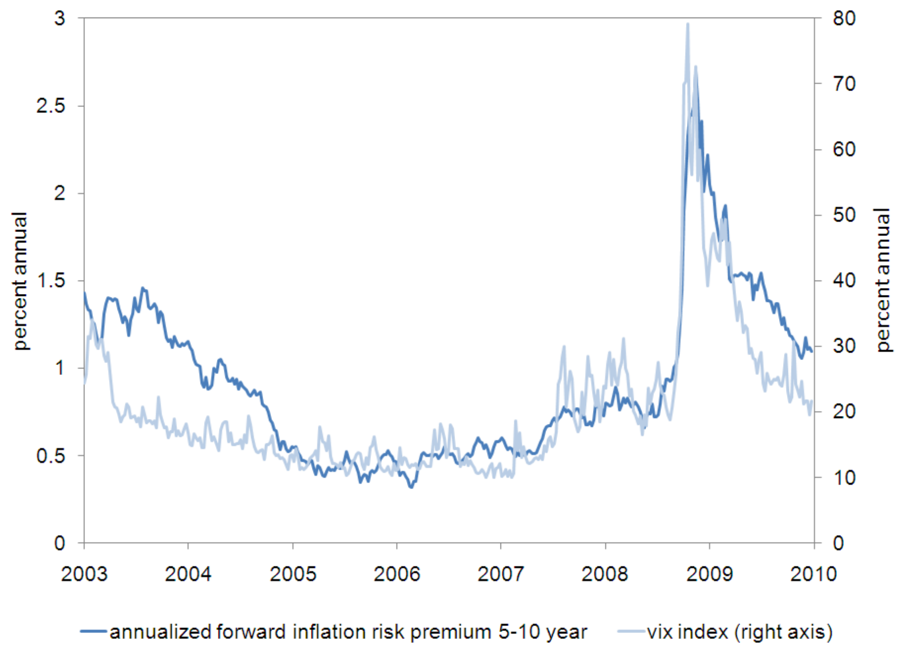

Figure 7: Inflation Risk Premium 5-10 Forward and Implied Equity Volatility VIX.

5 Robustness

5.1 Observability of second moments

A key difference between the approach taken in this paper, and most affine yield curve models presented in the literature is that we are using the estimated Garch variances and covariances as observables in the Kalman Filter. This gives rise to markedly different estimates of risk premia than in models where second moments are not entering as observables. This can be seen in Figure 8 where the inflation risk premium of our benchmark model is plotted together with the premium from two alternative models A and B. In both alternative A and B, second moments are unobservable, i.e. only the nominal and real yields enter into the observation equation of the Kalman Filter. Both models A and B have four factors. Model A has two inflation and two real factors,

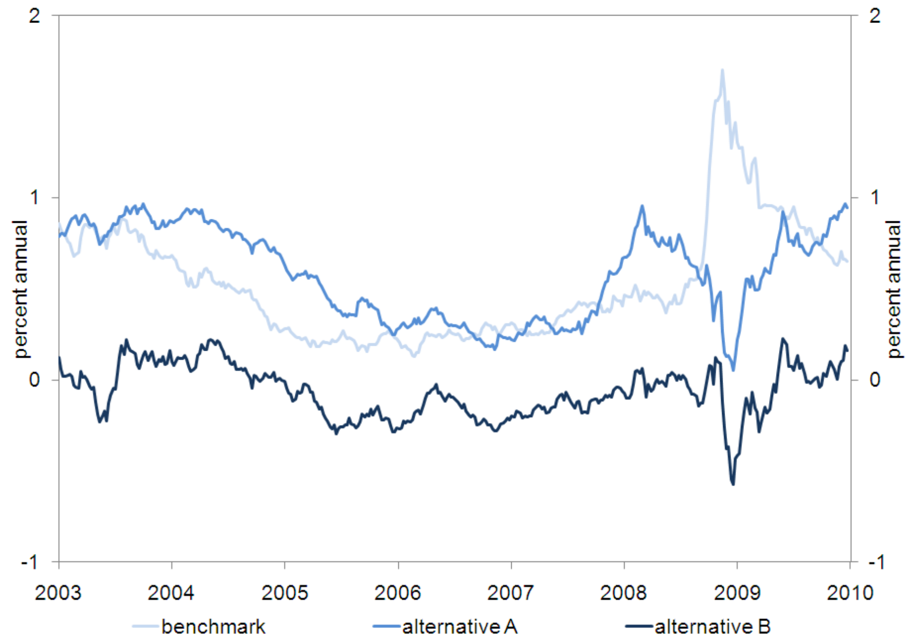

Figure 8: The Inflation Risk Premium from the Benchmark 5-Factor Model with Observable Inflation, and two Alternative Models A and B without Observable Inflation.

but no volatility factor, while model B has one inflation, two real, and one volatility factor. Figure 8 shows that alternatives A and B have less volatile inflation risk premia than the benchmark model, and both alternatives are pickig up a decline in the risk premium during the fall of 2008. Considering that the fall of 2008 was a period of unusual uncertainty, one would expect the inflation risk premium to increase, as is the case in our benchmark model. Both alternative models A and B change during the fall of 2008, but they both move in the wrong direction (the risk premium falls even though measures of uncertainty sharply increased). In our view, both alternative models A and B demonstrate that the observation of yields alone is not sufficient to precisely identify risk premia, which are tightly linked to time varying second moments.

5.2 Comparison to inflation and inflation expectations

In the benchmark model, actual inflation or inflation expectations do not enter as observable variables. The main motivation for this empirical strategy is that we want to extract the term structure of inflation expectations using only asset pricing data as input. That allows us to generate measures of inflation expectations that are complementary to the survey based expectations.

We estimated several specifications that allowed for either current inflation or survey inflation to enter the model as additional observables. However, the pricing performance is generally not improved in these models with additional observables.

What did improve was the correspondance between the inflation factor π and observed inflation, or between forward inflation and expected inflation. However, it is not clear that such alterations imply a better model. To illustrate this point, consider the plot in Figure 9. In this figure, the 10-year expected inflation from the term structure model is plotted together with the 5-year inflation forecast from the survey of professional forecasters (SPF). The plot shows that the SPF is more or less flat at 2.5%, without much variation. This long run inflation rate is slightly higher than the 2.1% implied inflation rate implied by Table 4.1 However, the inflation model implies substantial variations in expected inflation at the 5-year horizon around the financial crisis in 2008, as well as in the earlier period in 2003. The model implied inflation expectations thus differ substantially from the survey implied inflation forecasts.

The indexation of TIPS to the Consumption Price Index (CPI) introduces a predictable component in yield changes. This predictable component is sometimes called "carry", and a carry adjustment of yields is undoing this predictability. The interest and principal payments for TIPS are linked to the non-seasonally adjusted urban CPI with a three-month lag. The CPI is published every month. The daily reference index

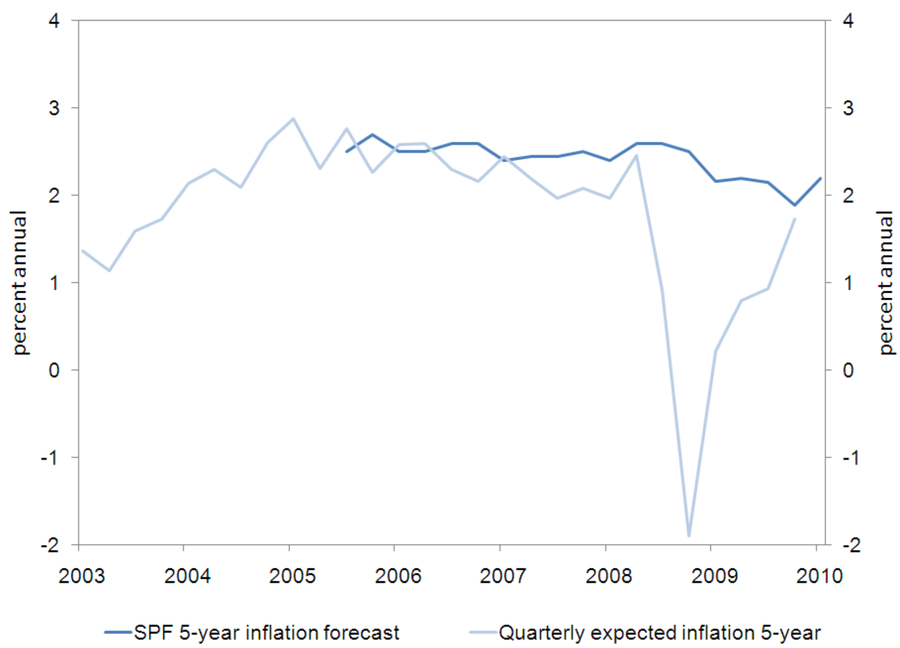

Figure 9: 5 Year Expected Inflation from the Survey of Professional Forecasters and from the Term Structure Model.

(DRI) for TIPS payoff and pricing calculation is computed based on the CPI values with two- and three-month lags (M2 and M3) as,

$$\begin{aligned}\text{Daily Reference Index} =& \text{Three-Month CPI Lag}\\ &+\tfrac{(\text{Today}-1)}{(\text{Number of Days in Month})}\,(\text{Two-Month CPI Lag} - \text{Three-Month CPI Lag}).\end{aligned}$$

TIPS principal is adjusted by multiplying the principal at issuance by the DRI at maturity and then dividing it by the DRI at issuance date. The adjusted principal is paid at maturity. The principal payment at maturity is

$$\$\ \text{Par Value}\times\tfrac{\text{Daily Reference Index}_{\text{maturity date}}}{\text{Daily Reference Index}_{\text{issuance date}}}$$

where the ratio of the two DRIs is often referred to as the index ratio.

We estimate specifications where the DRI is included in the observation equations (estimation results of such specifications is not reported here). We adjust for the carry effect by modeling the indexation lag explicitly. We do not find substantial differences in our estimates of the term structure of expected inflation, so we omit it from the current paper.

5.3 Testing for the number of factors

Our baseline specification has five factors: two factors of the real pricing kernel that capture the level and slope of the real term structure, two inflation factors to model inflation expectations, and one variance factor. However, we have estimated a number of specifications that allow for additional factors in the second moment matrix.

In Table 8, we report a test against an alternative model with eight factors: two real factors, two inflation factors, and three variance-covariance factors. We can see that the five factor model is rejected against the eight factor alternative at the 1%

level. So a model with additional factors in the variance-covariance matrix seems to perform better than a model with only one variance factor. However, we choose the 5-factor specification as the benchmark model as it is more parsimonious.

Table 8: Testing for the Number of Factors.

Model	Maximum Log Likelihood Value	Number of Parameters	Number of Observations
5-factor	-16316	37	365
8-factor	-72359	54	365
Tests	Chi Square	Significance	Degrees of Freedom
8-factor / 5-factor	112086	***	17

5.4 Structural break tests

Elsasser and Sack (2004) point out that the TIPS market was relatively illiquid for a number of years. The liquidity in the TIPS market biases real rates upwards, thus artificially compressing breakeven inflation rates. In order to see how the illiquidity might change our estimates, we fit the model separately before 2003 and since the beginning of 2003 (thus expanding the sample by three years. We use the end of 2002 as a break point as Elsasser and Sack argue that liquidity in the inflation protected market was comparable to the liquidity of the off-the-run Treasury market since then.

We do find a structural break at the beginning of 2003, as reported in Table 9. However, the filtered factors in the two separate sample periods is small. Importantly, the filtered factors do not change substantially. Our reason for using the 2003-2009 sample period in the baseline estimates—and not the longer 1999-2009 sample—is that the estimated unconditional mean of the inlation risk premium is markedly lower over the longer sample. The model does not allow us to discern to what extent this lower premium is due to the low liquidity in the TIPS market in the first few years, or whether it is due to the economic conditions during the earlier time period. As a result, we prefer to use the data since 2003 in our baseline specification.

Table 9: Testing for a Structural Break.

5-factor Model	Maximum Log Likelihood Value	Number of Parameters	Number of Observations
1999-2009	-27132	37	575
1999-2002	-4707.3	37	210
2003-2009	-16221	37	365
Tests	Chi Square	Significance	Degrees of Freedom
Structural Break in 2003	12407.4	***	37

6 Conclusion

We propose a novel methodology to extract the term structure of inflation expectations from the term structures of nominal and real interest rates. Our contribution is to fit an arbitrage free affine model not only to yields, but also their conditional variance-covariance matrix. We find that a five factor model with two real factors, two inflation factors, and one variance factor fits both first and second moments of the term structures well.

Our model can be updated weekly, making it suitable for market monitoring. We do find that there can be substantial differences between model implied inflation expectations, and breakeven inflation rates. These differences are highly correlated with market volatility measures such as the VIX equity implied volatility index. Intuitively, as implied volatility increases, risk premia increase, and breakevens tend to overpredict inflation expectations.

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