Endogenous Productivity and Development Accounting[1]

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Abstract

We model an environment with embodied technical change in which different vintages of capital with their different productivities coexist. A reduction in the cost of investment raises both the quantity and the productivity of capital simultaneously. The model induces a simple relationship between the relative price of investment goods and per capita income. Using cross-country data on the price of investment goods we find that the model does fairly well in quantitatively accounting for the observed dispersion in world income. For our baseline parameterization, the model generates 40-fold income gaps between the richest and poorest countries in our sample. The model also generates cross-country distributions of capital output ratios and productivity that track the data quite closely.

1 Introduction

Cross-country data reveals that the per capita incomes of the richest countries in the world exceed those in the poorest countries by a factor of 45. In this paper we formalize a model in which new, more productive vintages of capital coexist with older and less productive vintages. In such an environment, a lower relative price of investment induces a higher steady state capital stock as well as a higher level of average productivity. We quantify a calibrated version of the model using cross-country data on prices. The model can generate almost as much variation in cross-country relative income as is observed in the data. Under our baseline parameterization, the model generates 40-fold income gaps between the richest and poorest countries in the sample. The model also generates cross-country distributions of capital-output ratios and productivity that track the data reasonably well.

There is a large literature which examines the sources of differences in incomes across countries. There are two basic views. One school of thought holds that most of the differences in incomes across nations is due to differences in productivity across nations. The most well known expressions of this view are Hall and Jones (1999) and Parente and Prescott (1994, 1999). A second view holds that differences in measured inputs can account for a significant component of the differences in incomes (e.g., see Chari, Kehoe and McGrattan (1997), Mankiw, Romer and Weil (1992), Kumar and Russell (2002), Young (1995)). In related work Klenow and Rodriguez (1998) attempt a systematic and careful decomposition of the data and conclude that productivity differences account for upwards of 60% of the income dispersion across nations with measured inputs accounting for the balance.

The starting point for this paper is the well documented relationship between the relative price of investment and per capita income: poorer countries are also the countries where the price of capital goods (relative to the price of consumption goods) is higher (see, among others, Jones (1994), and Hall and Jones (1999)). However, the documented importance of productivity differences across countries suggests that the standard view of investment prices impacting income through their effect on capital accumulation (or more generally, measured inputs) can at best be a partial explanation for the observed income disparity across countries. A key goal of this paper is to formalize an environment wherein the price of investment affects the productivity of an economy over and above its standard effect on measured capital.

The main idea behind our work is that productivity and measured inputs are often determined jointly and they respond to the same set of economic decisions and incentives. In order to highlight this, we write down an exogenous growth model with embodied capital. We use a very simplified version of Hopenhayn (1992) in which investment occurs through entry. In every period, potential producers of intermediate goods face a choice of different types of capital (or machines) that they can invest in. Capital goods are tradeable and the available list of capital goods from which the intermediate goods producer chooses at any date includes all vintages of capital goods produced till that date. The labor productivity of the firm is pinned down by the technology vintage of the machine that the intermediate goods producer chooses. The productivity of the latest vintage of capital good (the frontier capital good) grows at an exogenous rate that is common to all countries. Different types of new capital goods are distinct in their productivities and price, with the newer/later vintages being more productive and more expensive. At any given time, the overall productivity of the economy reflects the mix of old and new capital as well as the mix of the types of new capital. Changes in the relative price of new capital induce changes in not only the stock of new capital but also in the average productivity of the economy due to the changing mix of new (high productivity) and older (low productivity) capital.

While the underlying structure of the model is complicated, we show that the behavior of the aggregate variables along a balanced growth path can be summarized by two variables: the average price of capital goods in the economy and the price of the latest capital good. Hence, these two prices serve as summary statistics for the model. We show that the per capita income gap across countries depends only on the cross-country gap in the price of frontier capital goods relative to the price of consumption. We also show that the productivity gap between countries depends on the cross-country gap in one relative price: the price of frontier capital goods relative to the average price of capital goods.

The main findings of the paper are quantitative. The model generates a steady-state distribution of relative incomes across countries as a function of the relative price of new capital. Using price data from the PWT dataset, we generate a cross-country income distribution using our model and compare its properties with the actual distribution. For our baseline parameterization, the model induces a cross-country distribution in which the per capita income of the richest countries exceeds that of the poorest countries in our sample by a factor of 30 which is almost the same as in the data. Moreover, the predicted relative income series tracks the actual relative income series quite closely, with the correlation between the two series being 0.75. We also use the model to generate a cross-country distribution of capital-output ratios. The correlation between the model generated capital-output ratios and the Hall and Jones capital-output series is 0.62. Lastly, we compute the productivities that would be measured by researchers if they imposed the Cobb-Douglas production function on data generated by our model. We find that the predicted productivity numbers measured from data generated by our model track the numbers reported in the Hall-Jones study reasonably well, with the correlation being 0.73. Based on these results, we consider the model to be a qualified success.

We also establish an ancillary analytical result. The pattern of equilibrium trade flows in the model is indeterminate. More specifically, the vector of equilibrium world capital goods prices is consistent with any pattern of capital trade across countries. Hence, the model can generate the two extreme equilibria – symmetric capital allocations and no-trade in capital goods – along with a continuum of capital allocations within this spectrum. Crucially, however, the relative income predictions of the model are independent of the precise trade flow patterns.

Since a key motivation for our work is the observed variation in the relative price of investment goods, one key observation is in order before we proceed. Hsieh and Klenow (2003) have argued that most of the variation in the relative price of investment goods in the PWT dataset is due to variations in the price of consumption across countries rather than variations in the price of investment goods. They interpret this result as suggesting that explanations of the world income dispersion that hinge on investment distortions in the form of import tariffs, taxes etc., are unlikely to be true. Instead, they argue the challenge is to explain the reasons for the low productivity of the investment goods sector in the poorer countries. Our model does not require a specific stand on whether the dispersion in the relative price of investment goods across countries is due to taxes or due to technology. All that is required for our results to go through is that there be observed variation in the cost of investment when expressed in terms of the domestic consumption good.

We would like to clarify that the reasons behind the cross-country variation of relative investment prices, while undoubtedly important to understand, are beyond the remit of this paper. Here, we simply ask whether the observed variation in prices, when passed through the lens of our model, can generate income variations along the lines observed in the data.

Our paper is related to previous work on models with vintage capital that were used to address cross-country data facts. Pessoa and Rob (2002) have a motivation which is very similar to our's. They write down a model of vintage capital with embodied technology and use it to show that given variations in investment distortions across countries create larger income differences than in the standard model. However, their model has a much richer but more complicated structure than our model. They choose a production function from a class of CES functions by estimating the parameters of the function. Their model allows firms to destroy old technology, adopt new technology, and to choose the quantity of the new capital to buy. This richness of structure comes at a significant cost of tractability and simplicity. Our model, while missing some of these features, provides a much simpler environment to solve and quantify. Gilchrist and Williams (2001) consider a model where technological change is embodied in new capital and at any point in time different vintages of capital coexist. However, in their model all steady state income differences are due to measured capital not productivity.[1]

Two other papers that are related to our work are Caselli and Wilson (2004) and Eaton and Kortum (2002). Caselli and Wilson note that there is huge variation in the composition of capital goods imports across countries. They then formalize a model in which capital composition in a country is linked to the productivity of different types of capital in that country. In their model the composition of capital provides a quality adjustment to the capital stock; hence it affects productivity. They use regressions to link these country-specific productivities of different types of capital to country characteristics such as education, property rights etc.. Using the estimated productivities they find that their model can account for a significantly larger share of the cross-country variation in relative incomes compared to the standard model with disembodied capital. There are two important differences between Caselli-Wilson and us. The first is an analytical difference. Caselli-Wilson focus on the productivity differences between different varieties of capital goods at a point in time while our focus is on productivity variations in capital goods over time; hence our focus is specifically on capital vintages while their's is on the cross-sectional capital composition at a point in time. The second differences concerns measurement. We measure cross-country differences in productivity by using the model dictated relationship between productivity and the price of investment goods. Caselli-Wilson measure cross-country productivity differences using regression estimates which link these to country characteristics. For both these reasons, we view our work as being complementary to the work of Caselli-Wilson since the papers emphasize different aspects of the data.

Eaton and Kortum (2002) develop a model with trade in capital goods. Their model predicts

[1] Our work is also related to Parente (1995) who develops a model of technology adoption. The key difference is that our framework formalizes environments with embodied technology while his work focuses on disembodied technology.

capital goods imports as a function of import prices of capital goods as well as other frictions to trade. They then use data on capital goods imports to derive a model implied series for the price of capital goods. Using this generated price series they show that the model can explain 25 percent of the cross-country variation in per capita income. The main difference of Eaton and Kortum's work from our's is that they do not focus on the cross-country differences in total factor productivity. While they allow productivity differences in the production technology for capital goods, these differences map into the price of capital goods, not the quality of the capital goods themselves. Thus, in their model a capital good which is cheaper to produce is used more. However, the output produced by a given combination of that capital good and other factors remains unaffected.

The rest of the paper is organized as follows: In the next section we lay out the model while Section 3 characterizes the steady state of the model. In Section 4 we describe the cross-country predictions of the model while in Section 5 we calibrate the model and present the quantitative results. The last section concludes.

2 Model

We consider a world economy with many open economies. We first describe one of these open economies and then proceed to discuss the cross-country implications of the model.

Time is discrete $t = 0, 1, ...$ The environment is characterized by perfect foresight: all agents know past, present, and future realizations of exogenous variables with probability one. At any time t, the economy is inhabited by L_t identical households who consume a final good and supply labor inelastically. We let the final good be the numeraire good so that all prices within an economy are in terms of the final good.

The final good is produced by a perfectly competitive representative firm by combining a list of differentiated intermediate goods. Each intermediate good is provided by a monopolistically competitive firm. Intermediate goods are produced by combining labor input with a number of capital goods (which we call "machines").

Investment is realized through entry in the intermediate goods sector. Entering firms have a menu of investment options. They can either invest in the state of the art machines which embodies the frontier technology available; else they can invest in any older machines with the corresponding vintage technology. The "technology" of the machine determines the labor productivity of the firm. Machines with superior technology come at a higher cost. Once a machine is bought/installed, its productivity remains fixed for the duration of the life of the machine. Lastly, productivity of the frontier technology is assumed to grow at an exogenous rate which is common to all economies of the world.

Capital goods are produced by a sector of perfectly competitive firms. They are also the only tradeable goods in the economy. We also assume that trade is balanced in every period. Differences in the capital good production technology are the only source of variation across countries.

2.1 Households

The representative household maximizes the present discounted value of lifetime utility

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta}$$

subject to

$$c_t + q_t b_t \leq w_t + d_t + b_{t-1}$$

for all $t \geq 0$, where $\theta > 0$ and c_t is consumption of the representative household and b_t are one-period bonds contracted at date t that pay one unit of the final good next period.[2] Bonds are sold at discount at price q_t. Wages are given by w_t, and d_t are dividends from all firms. The representative household inelastically supplies one unit of labor every period.

[2] Under our assumption of balanced trade, households do not have access to international capital.

The first order condition for the household problem leads to the standard Euler equation

$$q_t = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\theta} \tag{1}$$

which prices the bond. Let $q_t^j = q_t q_{t+1} ... q_{t+j}$ for $j \geq 1$.

2.2 Final Goods Sector

The final good is produced by combining a set Ω_t of distinct intermediate goods according to

$$Y_t = \left[\int_{\Omega_t} \left[y_t \left(\omega \right) \right]^{\rho} d\omega \right]^{\frac{1}{\rho}}$$

where $0 < \rho < 1$.

A perfectly competitive final good firm chooses inputs $y_t(\omega)$ to maximize profits

$$\pi_t^f = Y_t - \int_{\Omega_t} p_t(\omega) y_t(\omega) d\omega$$

subject to the posted prices, $p_t(\omega)$, for each intermediate good $\omega \in \Omega_t$.

We index intermediate goods by their technology as given by their labor productivity $\varphi \in \Re^+$. This turns out to be convenient as technology differences are the source of all the relevant firm heterogeneity in the model. In other words, all goods/firms ω which share the same technology φ are indeed identical in their price and production decisions.

Let $M_t(\varphi)$ be the measure of goods/firms with technology φ. We can then rewrite the final good production function as

$$Y_t = \left[\int \left[y_t \left(\varphi \right) \right]^{\rho} M_t \left(\varphi \right) d\varphi \right]^{\frac{1}{\rho}} \tag{2}$$

and the implied demand

$$y_t \left(\varphi \right) = Y_t \left[p_t \left(\varphi \right) \right]^{-\sigma} \tag{3}$$

where $\sigma = \frac{1}{1-\rho} > 1$ denotes the elasticity of demand for each final good.

Since this sector is perfectly competitive, the representative final good firm must be making zero profits. Hence, at each date we have

$$Y_t - \int p_t(\varphi) y_t(\varphi) M_t(\varphi)\, d\varphi = 0,$$

and substituting in (3)

$$\int [p_t(\varphi)]^{1-\sigma} M_t(\varphi)\, d\varphi = 1. \tag{4}$$

2.3 Intermediate goods firms

Intermediate goods firms in this economy produce output using a production technology that is linear in labor. Specifically, the production function is:

$$y_t(\varphi) = \varphi l_t(\varphi)$$

where φ is the productivity of the firm and $l_t(\varphi)$ its labor demand.[3] Hence, higher productivity is labor saving in that it lowers the labor required to produce the same unit of output.

Intermediate goods firms are monopolistically competitive and maximize profits at every date t by choosing the price of their good subject to the inverse demand function (equation (3)). Profits of firm φ at date t are given by

$$\pi_t(\varphi) = p_t(\varphi) y_t(\varphi) - w_t l_t(\varphi)$$

where w_t is the wage rate. The intermediate firm's problem implies an optimal pricing rule given by

$$p_t(\varphi) = \frac{w_t}{\rho\varphi}. \tag{5}$$

Note that the pricing rule implies that higher productivity firms will charge a lower price and thus have higher sales.

[3] We describe intermediate firms by their technology for expositional convenience. But it is important to keep in mind that every firm produces a distinct good even if they share the technology level.

Using the optimal pricing rule (5), it is straightforward to check that

$$\pi_t(\varphi) = \frac{1}{\sigma} p_t(\varphi) y_t(\varphi)$$

so profits are a share $\frac{1}{\sigma}$ of revenues. Note that relative profits are scaled by the level technology:

$$\frac{\pi_t(\varphi)}{\pi_t(\varphi')} = \left(\frac{\varphi}{\varphi'}\right)^{\sigma-1}.$$

Hence higher productivity firms have higher profits.

2.4 Entry and Exit of Intermediate Good Firms

At every date there is a infinite pool of entrants. An entrant into the industry needs to purchase a number of capital goods (or machines) in order to start producing a new intermediate good. Once the initial start-up investment is made, production only requires labor as given in the production function above.

There are many different vintages of capital goods to choose from: the entering firm's investment decision determines its labor productivity φ. At every date there is a state-of-art or frontier machine which is embodied with labor productivity φ_t. We assume that the productivity of the frontier machine evolves at an exogenous rate $\gamma > 1$,

$$\frac{\varphi_{t+1}}{\varphi_t} = \gamma. \tag{6}$$

In addition to the frontier machine, at every date there are machines of vintage $t-1, t-2...$ A machine of vintage $t-j$ is embodied with labor productivity φ_{t-j}, i.e., the corresponding frontier's machine productivity at date $t-j$.[4]

We also assume that every period there is an exogenous exit rate δ of existing intermediate goods firms. Specifically, at the end of each period a fraction of δ of the existing stock of machines

[4] For simplicity, all investment must be made on the same type of capital good, i.e., it is not possible to combine machines of different vintage to start up production.

being used by intermediate goods firms in that period breaks down. Let $N_t(\varphi)$ be the measure of entrants who invest in a machine with embodied technology φ; the resulting law of motion for $M_t(\varphi)$ is then

$$M_t(\varphi) = N_t(\varphi) + (1-\delta) M_{t-1}(\varphi).$$

We use N_t and M_t for total entrants and active producers at date t.

Let $v_t(\varphi)$ be the present value of an intermediate good firm with productivity φ operating at date t, net of entry costs,

$$v_t(\varphi) = \sum_{j=0}^{\infty} (1-\delta)^j q_t^j \pi_{t+j}(\varphi).$$

It is assumed that every intermediate good firm is owned by the representative household and hence profits in future periods are discounted according to q_t^j.

We assume that, independently of which capital goods are used, the number of capital goods needed to start up production is proportional to the size of the economy.[5] Let F_t^j be the total cost of a machine of vintage $t-j$, given by $F_t^j = f_t^j L_t$ for all t, where f_t^j is the price of a machine. This formulation equates the total number of final good producers, M_t, to the number of active machines per capita.

An entering firm at date t chooses the capital good of vintage $t-j$ which solves

$$\max_{j\geq 0} \left\{ v_t\left(\varphi_{t-j}\right) - F_t^j \right\}.$$

There will be positive entry in the intermediate good sector as long as it is profitable using *any* capital good

$$\max_{j\geq 0} \left\{ v_t\left(\varphi_{t-j}\right) - F_t^j \right\} \geq 0.$$

[5] This assumption formalizes the idea that a larger economy with more labor needs machines with bigger capacity (or equivalently, it needs a larger machine). Hence, the same productivity machine costs proportionately more in an economy with a larger labor force. This assumption ensures that the model does not generate any scale effects on development.

Entry will continue until there are no positive rents left from entry. Thus, the free entry condition is that

$$\max_{j\geq 0}\left\{v_t\left(\varphi_{t-j}\right)-F_t^j\right\}\leq 0 \tag{7}$$

with strict equality if there is positive entry, $N_t > 0$. We can write a free entry condition for each $j \geq 0$,

$$v_t\left(\varphi_{t-j}\right)\leq F_t^j \tag{8}$$

with strict equality if there is positive entry with a machine of vintage $t-j$, i.e., $N_t\left(\varphi_{t-j}\right) > 0$.

We will use a vintage notation as follows

$$M_t^j = M_t\left(\varphi_{t-j}\right)$$

and similarly for N_t^j, p_t^j, and π_t^j.

2.5 Capital Goods

Capital goods are the only tradeable goods in the economy. Each capital good producer takes as given the world prices for capital goods, denoted ϕ_t^j. We abstract from trade frictions, and therefore we have the following law of one price

$$f_t^j = \varepsilon_t \phi_t^j$$

for all $j \geq 0$, where ε_t is the real exchange rate defined in terms of the final good.

Capital goods are provided by perfectly competitive firms. In order to produce a machine of vintage $t-j$ at date t, the representative capital good firm uses $g_t^j\left(x_t^j\right) > 0$ units of the final good, where g_t^j is a continuous and increasing function and x_t^j is the local production of capital goods of such vintage. The assumption of an upward sloping cost curve reflects the presence of some factor in limited supply.

Perfect competition equates price to marginal cost

$$f_t^j = g_t^j\left(x_t^j\right) \tag{9}$$

if $x_t^j > 0$. Net exports of vintage $t-j$ capital good are $\left(x_t^j - N_t^j L_t\right) f_t^j$.

We want to guarantee that all available capital goods are produced in all countries along the balanced growth path. For this we postulate that $g_t^j(0)$ is low enough such that $v_t\left(\varphi_{t-j}\right)/L_t > g_t^j(0)$ for all $j \geq 0$ along the balanced growth path. This greatly simplifies the analysis at little cost: all machines have a positive exit rate $\delta > 0$, so the gross entry rate can be positive yet small enough for machines of older vintages, so that there is a positive exit rate.

2.6 Market Clearing Conditions and Equilibrium Definition

Before defining a competitive equilibrium, we need to state the market clearing conditions. First, the labor market requires that we have

$$\int l_t(\varphi) M_t(\varphi)\, d\varphi = L_t \qquad \text{for all } t. \tag{10}$$

Second, balanced trade implies that

$$\sum_{j=0}^{\infty} \left(x_t^j - N_t^j L_t\right) f_t^j = 0 \tag{11}$$

for all t. Finally, we can use equation (11) to write the resource constraint for this economy is

$$c_t + \sum_{j=0}^{\infty} N_t^j f_t^j = Y_t/L_t. \tag{12}$$

Definition 1 *A small open economy equilibrium* Γ *is a sequence of prices*

$$\left\{\left\{p_t^j, f_t^j\right\}_{j\geq 0}, q_t, w_t, \varepsilon_t\right\}_{t\geq 0}$$

and quantities

$$\left\{\left\{M_t^j, N_t^j, x_t^j, y_t^j, l_t^j\right\}_{j\geq 0}, c_t, Y_t\right\}_{t\geq 0}$$

such that for all $t \geq 0$

1. *The household problem is solved, i.e., (1) holds.*

2. *All firms maximize profits.*

3. *The free entry conditions (8) are satisfied.*

4. *All markets clear.*

2.7 A World Equilibrium

Let C be the set of countries in the economy. In the world equilibrium, each country constitutes a small open economy equilibrium and world prices clear the international market of each capital good.

Definition 2 *A world equilibrium is a system of small open economy equilibria* $\{\Gamma_c : c \in C\}$ *and world prices* $\left\{\left\{\phi_t^j\right\}_{j\geq 0}\right\}_{t\geq 0}$ *such that*

$$\sum_{c\in C}\left(x_{ct}^j - N_{ct}^j L_{ct}\right) = 0 \tag{13}$$

for all j *and* t.

Of course, only $N-1$ prices are pinned down in equilibrium as one country's consumption acts as the numeraire in the world markets.

2.8 Solving for Equilibrium

We start by noting that zero profits for final goods firms implies that

$$Y_t = \int p_t(\varphi) y_t(\varphi) M_t(\varphi)\, d\varphi.$$

Substituting the production technology for intermediate goods and the optimal pricing equation (5) gives

$$Y_t = \frac{1}{\rho} w_t L_t.$$

Hence the wage is proportional the income per person in this economy — which we denote by y_t.

Next, we can solve for equilibrium wages by substituting the optimal intermediate goods pricing equation (5) into equation (4)

$$\begin{aligned} 1 &= \left[\int \left(\frac{w_t}{\rho\varphi}\right)^{1-\sigma} M_t(\varphi)\, d\varphi\right], \\ w_t^{\sigma-1} &= \rho^{\sigma-1}\left[\int \varphi^{\sigma-1} M_t(\varphi)\, d\varphi\right], \end{aligned}$$

and factoring out M_t,

$$y_t = \tilde{\varphi}_t M_t^{\frac{1}{\sigma-1}} \tag{14}$$

where we define the average technology $\tilde{\varphi}_t$ at date t as

$$\tilde{\varphi}_t = \left[\int \varphi^{\sigma-1}\frac{M_t(\varphi)}{M_t} d\varphi\right]^{\frac{1}{\sigma-1}}.$$

Expression (14) determines income per capita.

We can use equations (3) and (5) to rewrite revenues of intermediate goods firms as

$$p_t(\varphi)\, y_t(\varphi) = \varphi^{\sigma-1} y_t^{2-\sigma} L_t.$$

Substituting this expression for revenues into the expression for intermediate firms' profits gives

$$\pi_t(\varphi) = (1-\rho)\,\varphi^{\sigma-1} y_t^{2-\sigma} L_t. \tag{15}$$

Since the free entry conditions rule out positive rents from entry, intermediate good firms use their profits to finance the initial investment. Hence, $\frac{1}{\sigma}$, which is the ratio of profits to revenues, can be equated to the share of capital in this model.

Using equation (14) one can re-arrange the expression for profits and write it as

$$\pi_t(\varphi) = (1-\rho)\,\varphi^{\sigma-1}\tilde{\varphi}_t^{1-\sigma}\left(\tilde{\varphi}_t M_t^{\frac{2-\sigma}{\sigma-1}} L_t\right).$$

In order to have a bounded economy, we need profits to fall with entry. Hence we impose the restriction $\sigma > 2$. The term in parenthesis is also the ratio of output to machines, y_t/M_t. Note that if $\sigma < 2$, this ratio would be increasing in the stock of machines.

3 Balanced Growth Path

We now characterize a steady state balanced growth path for this economy. In particular, we look for paths along which M_t, Y_t, c_t and $\left\{f_t^j\right\}_{j\geq 0}$ grow at a constant rate. In the following we shall use γ_j to denote the constant, steady state rate of growth of variable $j = M, Y, y, L...$ Recall that both the frontier technology φ_t and the labor force L_t grow at an exogenously given constant growth rate.

Another possible source of growth is a downward trend in the cost of capital goods. We abstract from this possibility by assuming that the price of a capital good of a certain age is constant over time along a balanced growth path: $f_t^j = f_{t+1}^j$ for all j. Hence, f^j is independent of time. Note that this assumption does not imply that the price of a capital good of a given vintage is constant. As we show below, in equilibrium the price of a vintage declines as it gets older: $f_t^j > f_{t+1}^{j+1}$.[6]

We now proceed to derive several results for the balanced growth path. First, along the balanced growth path, the price of the bonds will be constant,

$$\tilde{q} = \beta\gamma_c^{-\theta}$$

as derived from (1).

Second, we want to solve for the net present value of profits at any given date. Using our characterization of profits, it follows that along the balanced growth path

$$\frac{\pi_{t+1}(\varphi)}{\pi_t(\varphi)} = \frac{\gamma_y\gamma_L}{\gamma_M\gamma_{\tilde{\varphi}}^{\sigma-1}}.$$

Therefore we can write the free entry condition for vintage j (8) as

$$\pi_t\left(\varphi_{t-j}\right)\sum_{i=0}^{\infty}\left((1-\delta)\,\tilde{q}\frac{\gamma_y\gamma_L}{\gamma_M\gamma_{\tilde{\varphi}}^{\sigma-1}}\right)^i \leq F_t^j.$$

[6] The assumption, of course, is in terms of the process underlying the cost functions g_t^j. Like any model with investment and consumption sectors, the investment price is only constant if the productivity growth rates in both sectors satisfy a point condition.

It suffices to assume a high value of δ to guarantee that the left hand side is finite. The CES demand specification implies that

$$\frac{v_t\left(\varphi_{t-j}\right)}{v_t\left(\varphi_t\right)}=\frac{\pi_t\left(\varphi_{t-j}\right)}{\pi_t\left(\varphi_t\right)}=\left(\frac{\varphi_{t-j}}{\varphi_t}\right)^{\sigma-1}.$$

With positive entry in every vintage, we then have

$$\frac{f_t^j}{f_t^0}=\left(\frac{\varphi_{t-j}}{\varphi_t}\right)^{\sigma-1} \tag{16}$$

from combining any two free entry conditions (8) with strict equality. Condition (16) is key to this paper. As long as there is positive entry, the relative price of two capital good of different vintage is given by the technology path. Hence, in equilibrium, capital goods price inherit the balanced growth path properties of technology. Specifically, the price of a capital good is falling at rate $\gamma^{1-\sigma}$.

Condition (16) also helps us to solve for growth rates. From the expression for income (14) we get

$$\gamma_y=\gamma_{\tilde{\varphi}}\gamma_M^{\frac{1}{\sigma-1}}.$$

Recalling that $y=Y/L$, it trivially follows that $\tilde{\varphi}_t$ must be growing at a constant rate if both y and M grow at constant rates. The binding entry condition (8) for the same capital good taken across two adjacent time periods gives

$$\frac{v_{t+1}\left(\varphi_{t-j}\right)}{v_t\left(\varphi_{t-j}\right)}=\frac{F_{t+1}^{j+1}}{F_t^j}.$$

The right hand side of this expression is the ratio of the entry cost of the same capital good across period. This can be written as

$$F_{t+1}^{j+1}/F_t^j=\gamma_L f_{t+1}^{j+1}/f_t^j.$$

Since vintage prices are constant over time, i.e., $f_t^0 = f_{t+1}^0$, it follows that

$$\begin{aligned} F_{t+1}^{j+1}/F_t^j &= \gamma_L \left(\frac{f_{t+1}^{j+1}}{f_{t+1}^0} \right) \left(\frac{f_t^0}{f_t^j} \right) \\ &= \gamma_L \left(\frac{\varphi_t}{\varphi_{t+1}} \right)^{\sigma-1} \\ &= \gamma_L \gamma^{1-\sigma}. \end{aligned}$$

Following the same steps that were followed to derive (16) we can establish that

$$\frac{v_{t+1}(\varphi)}{v_t(\varphi)} = \frac{\gamma_y \gamma_L}{\gamma_M \gamma_{\tilde{\varphi}}^{\sigma-1}}.$$

Combining both these results yields

$$\frac{\gamma_y \gamma_L}{\gamma_M \gamma_{\tilde{\varphi}}^{\sigma-1}} = \gamma_L \gamma^{1-\sigma}.$$

Rearranging and using $\gamma_y = \gamma_{\tilde{\varphi}} \gamma_M^{\frac{1}{\sigma-1}}$, we get

$$\gamma_y = \gamma^{\frac{\sigma-1}{\sigma-2}}.$$

From the resource constraint (12), it follows that the ratio M_t/y_t must be constant. Otherwise, either consumption contracts or explodes as a share of output. Hence, $\gamma_M = \gamma_y$. This implies that $\tilde{\varphi}_t$ grows at rate γ along a balanced growth path. Since $\tilde{\varphi}$ and φ grow at the same rate it follows that along a balanced growth path the average technology is at a fixed distance from the technological frontier.

Finally, we show that the distribution of capital vintages is constant along a balanced growth path. From the definition of $\tilde{\varphi}_t$, we have

$$\tilde{\varphi}_t^{\sigma-1} = \varphi_t^{\sigma-1} \sum_{j=0}^{\infty} \left(\frac{M_t^j}{M_t} \right) \gamma^{(1-\sigma)j}.$$

The discussion above concluded that $\tilde{\varphi}_t/\varphi_t$ is constant along a balanced growth path. It follows that the distribution of vintages $\left\{ M_t^j \right\}_{j \geq 0}$ is invariant once scaled by total capital M_t, i.e.,

$$\mu^j \equiv \frac{M_t^j}{M_t} = \frac{M_{t+1}^j}{M_{t+1}}.$$

Otherwise, the sum $\sum_{j=0}^{\infty}\left(\frac{M_t^j}{M_t}\right)\gamma^{(1-\sigma)j}$ would not be constant.

Recapping, we have established a key relationship between capital goods prices as captured by equation (16). We then solved for the growth rates of output, capital and average productivity. We showed that these growth rates were functions of the exogenous growth rate of the technology frontier and do not depend on the cost of investment. Crucially, we have said nothing about the actual distribution of capital goods $\left\{\mu^j\right\}_{j\geq 0}$ other than it is invariant along the balanced growth path.

4 Cross-country comparisons

We now turn to cross-country steady state comparisons implied by the model. We posit that differences in the capital good production technology as the sole source of cross-country variation. For everything else, all countries are identical.[7]

Countries can have different capital good distributions $\left\{\mu^j\right\}_{j\geq 0}$ as well as be different in the number of machines M. Both map into cross-country variation in income. To see this, recall that income per capita is given by (14)

$$y_t = \tilde{\varphi}_t M_t^{\frac{1}{\sigma-1}}.$$

Different distributions $\left\{\mu^j\right\}_{j\geq 0}$ shift the average productivity term $\tilde{\varphi}_t$, depending on whether the majority of capital goods are close to the technological frontier or not.

At this point, our model is a complex one. In order to solve for cross-country income differences, it seems we would have to first posit a theory of the cost of capital. A quantitative evaluation appears a daunting task: it appears to be necessary to know the distribution of labor productivity across existing firms, as well as have access to disaggregated data on capital good prices. However, we show below that all aggregate variables in the model can be expressed as functions of only two prices: the average price of capital goods and the price of the frontier capital good.

[7] From the discussion in the previous section, it also follows that all countries share the same growth rate.

The remainder of this section proceeds as follows. First we prove our claim that the price of the average and the frontier capital goods are summary statistics for aggregate income and productivity. Second we solve for cross-country income differences, highlighting the variation in both average productivity and capital intensity as a function of both the average and frontier's price. We then show that trade flows are not determined in equilibrium, but income is. Lastly we include a closed economy example.

4.1 Just Two Moments

Consider a machine whose embodied technology is equal to the average productivity of the economy at the present date, $\tilde{\varphi}_t$. We will call this artificial construct the "average" machine. We deduce a price for the average machine, denoted $\tilde{f}_t$, from the (fictitious) free entry condition

$$\tilde{F}_t = v_t\left(\tilde{\varphi}_t\right)$$

where $\tilde{F}_t = \tilde{f}_t L_t$.

Like in the computation of the relative price of capital good vintages (16), we can combine the entry condition of the average machine with the frontier machine,

$$\left(\frac{\tilde{\varphi}_t}{\varphi_t}\right)^{\sigma-1} = \frac{\tilde{f}_t}{f_t^0}. \tag{17}$$

Expression (17) allows us to solve for the price of the average machine as the central first moment of the capital good price distribution,

$$\begin{aligned}\tilde{f}_t &= \left(\frac{\tilde{\varphi}_t}{\varphi_t}\right)^{\sigma-1} f_t^0 \\ &= \sum_{j=0}^{\infty} \mu^j \left(\frac{\varphi_{t-j}}{\varphi_t}\right)^{\sigma-1} f_t^0 \\ &= \sum_{j=0}^{\infty} \mu^j f_t^j.\end{aligned}$$

Hence, the price of the average machine is the average price among existing machines. Note that the weights are given by the vintage distribution $\mu^j = \frac{M_t^j}{M_t}$.

Now that we view the price of the average machine as just the average price of machines, the relationship (17) is quite revealing. Average productivity is just a function of the ratio of the average to the frontier capital good price. That is, we only need two moments of the capital good price distribution: the average price $\tilde{f}_t$ and the maximum f_t^0, which also corresponds to the price of the frontier machine.

What about the number of machines and income per capita? It turns out these can also be expressed as functions of f_t^0 and $\tilde{f}_t$. The free entry condition for the frontier machine, along the balanced growth path, can be written as

$$\pi_t(\varphi_t) \sum_{i=0}^{\infty} \left((1-\delta)\, \tilde{q} \frac{\gamma_y \gamma_L}{\gamma_M \gamma_{\tilde{\varphi}}^{\sigma-1}} \right)^i = F_t^0.$$

Using some algebra on the profits we get

$$\left(\frac{\varphi_t}{\tilde{\varphi}_t} \right)^{\sigma-1} \left(\frac{y_t}{M_t} \right) A = f_t^0$$

where $A = (1-\rho) \sum_{i=0}^{\infty} \left((1-\delta)\, \tilde{q} \frac{\gamma_y \gamma_L}{\gamma_M \gamma_{\tilde{\varphi}}^{\sigma-1}} \right)^i$. Using equation (17), it follows that the output-to-machine ratio is

$$\frac{y_t}{M_t} = A^{-1} \tilde{f}_t. \tag{18}$$

Since income per capita is given by

$$y_t = \tilde{\varphi}_t M_t^{\frac{1}{\sigma-1}}$$

it is trivial to solve for M_t and y_t as functions of $\left\{ \tilde{f}_t, f_t^0 \right\}$ and parameters.

4.2 Income Differences

A central variable of interest for our cross-country comparisons is income per capita (y_t). We seek to express these in terms of the differences in $\left\{ \tilde{f}_t, f_t^0 \right\}$. In the following, we shall compare two countries by following the notational convention of denoting the second country variables with primes.

Since the process for φ_t is common, equation (17) implies that

$$\frac{\tilde{\varphi}_t}{\tilde{\varphi}_t'} = \left(\frac{f_t^{0\prime}}{\tilde{f}_t'}\frac{\tilde{f}_t}{f_t^0}\right)^{\frac{1}{\sigma-1}} \tag{19}$$

which shows that the productivity gap between countries depends on the difference in the relative cost of frontier to average machines across countries. The higher the relative price of frontier machines the lower is the relative productivity level of the country.

The free entry conditions for the notional average machine at home and abroad are given by

$$v_t\left(\tilde{\varphi}_t\right) = \tilde{F}_t,$$
$$v_t\left(\tilde{\varphi}_t'\right) = \tilde{F}_t'.$$

Combining the two conditions gives

$$\frac{\tilde{\varphi}_t}{\tilde{\varphi}_t'}\left(\frac{M_t}{M_t'}\right)^{\frac{2-\sigma}{\sigma-1}} = \frac{\tilde{f}_t}{\tilde{f}_t'}. \tag{20}$$

Substituting equation (19) in (20) then gives

$$\frac{M_t}{M_t'} = \left(\frac{f_t^{0\prime}}{f_t^0}\right)^{\frac{1}{\sigma-2}}\left(\frac{\tilde{f}_t'}{\tilde{f}_t}\right) \tag{21}$$

This expression gives the ratio of machines at any given date along a balanced growth path. The ratio of machines depends in an obvious way on the cost of investing in both old and new machines – the higher the cost of a new machine (both f_t^0 and $\tilde{f}_t$) the lower is M_t/M_t'.

Next, recall that per capita output is given by $y_t = \tilde{\varphi}_t M_t^{\frac{1}{\sigma-1}}$. Hence,

$$\frac{y_t}{y_t'} = \frac{\tilde{\varphi}_t}{\tilde{\varphi}_t'}\left(\frac{M_t}{M_t'}\right)^{\frac{1}{\sigma-1}}.$$

Using equations (19) and (21), this can be rewritten as

$$\frac{y_t}{y_t'} = \left(\frac{f_t^{0\prime}}{f_t^0}\right)^{\frac{1}{\sigma-2}}. \tag{22}$$

Hence, the income gap across countries depends on the relative cost of frontier machines. In particular, the higher the relative cost of the frontier machine in a country the lower is its relative per capita income.[8]

[8] It is instructive to note that the ratio of per capita steady state incomes can also be written as $\frac{y}{y'} =$

4.3 Indeterminacy of Trade Flows

In our model the cross-country income distribution, real exchange rates and the pattern of production are locally determined along the balanced growth path. However, the trade flows are undetermined and, by extension, so is the composition of the stock of machines in a given country. This is not surprising given that the relative price of two capital goods is pinned down by their embodied labor productivity — see equation (16). Hence, given a production pattern, machines can be rearranged across countries without changing the cross-country distribution of income or real exchange rates.

Let us start by showing that no trade is a world equilibrium despite the fact that all capital goods are frictionlessly tradeable and countries may differ in their capital good production technology. Note that the balanced trade condition (11) and the world market clearing conditions are trivially satisfied by setting $N_{ct}^j = x_{ct}^j$. The vector of capital good prices in country c, $\left\{f_{ct}^j\right\}_{j\geq 0}$, then determines jointly the production pattern, $\left\{x_{ct}^j\right\}_{j\geq 0}$ and, through the balanced growth path conditions on the law of motion of each capital good vintage, $\left\{M_{ct}^j\right\}_{j\geq 0}$. The entry conditions for each capital good provide the final set of equilibrium conditions. The real exchange rate is then given by the ratio of the frontier capital good prices across two given countries and (16) implies that the law of one price holds for all tradeables.

Next we show how to construct different world equilibria with different trade flows and composition of capital but the same incomes, production location, and real exchange rates as in the no-trade equilibrium.

$\left(\frac{\varphi}{\varphi'}\right)^{\frac{\sigma-1}{\sigma-2}}\left(\frac{M/Y}{M'/Y'}\right)^{\frac{1}{\sigma-2}}\left(\frac{L}{L'}\right)^{\frac{1}{\sigma-2}}$. This expression looks very similar to the standard expression for the income ratio under the Solow model with a Cobb-Douglas production technology. The only difference is that in our case the last two terms on the right hand side (which are measured inputs) are raised to the power $(\sigma-2)^{-1}$ while in the Solow model they are raised to a power which is the ratio of the capital share to the labor share. Hence a $\sigma = 2.5$ would generate a fit for our model analogous to the fit of the neoclassical model with a capital share of 2/3.

There are two steps. First we show that the balanced trade condition only depends on the actual income level and not on the precise composition of capital. Second we show how to re-organize capital goods around two countries.[9]

First Step. Take the balanced trade condition (11) for a given country, and divide by f_c^0. We can write it as

$$\sum_{j=0}^{\infty} \frac{f_c^j}{f_c^0} N_c^j = \sum_{j=0}^{\infty} \frac{f_c^j}{f_c^0} x_c^j$$

Use the relative price equation to write

$$\sum_{j=0}^{\infty} \varphi_{t-j}^{\sigma-1} N_c^j = \sum_{j=0}^{\infty} \varphi_{t-j}^{\sigma-1} x_c^j.$$

The balanced growth conditions make $\sum_{j=0}^{\infty} \varphi_{t-j}^{\sigma-1} N_c^j$ proportional to output. To see this, recall that the "law of motion" for $\tilde{\varphi}$ is given by

$$\tilde{\varphi}_t^{\sigma-1} = (1-\delta) \frac{M_{t-1}}{M_t} \tilde{\varphi}_{t-1}^{\sigma-1} + \frac{\sum_{j=0}^{\infty} \varphi_{t-j}^{\sigma-1} N_t^j}{M_t}.$$

Using the fact that M and $\tilde{\varphi}$ are growing at constant rates along a balanced growth path, the above can be rewritten as

$$\begin{aligned} \sum_{j=0}^{\infty} \varphi_{t-j}^{\sigma-1} N_t^j &= \mathcal{G} \tilde{\varphi}_t^{\sigma-1} M_t \\ &= \mathcal{G} w_t^{\sigma-1}, \end{aligned}$$

where $\mathcal{G}$ is a constant.

Hence, any (balanced growth path) allocation that delivers the same income as in the closed economy will satisfy the balanced trade conditions (11).

Second Step. Consider just two countries A and B. Take the closed economy allocations. We will consider perturbing the distribution of capital M^j. For small deviations, we can actually look at the market clearing conditions (13) as

$$M_{At}^j + M_{Bt}^j = \bar{M}^j$$

[9] For expositional simplicity, we set both countries to have a labor workforce of 1.

where $\bar{M}^j$ is the world use of capital good of age j. The perturbation will leave $\bar{M}^j$ constant for all j. Since there is positive entry everywhere, changes in N_t^j can actually implement small changes in $\bar{M}^j$.

Write income (recall equation 14) using the approximation to the world market clearing conditions as

$$\begin{aligned} y_{At}^{\sigma-1} &= \sum_{j=0} \varphi_{t-j}^{\sigma-1} M_{At}^j \\ &= \sum_{j=0} \varphi_{t-j}^{\sigma-1} \left(\bar{M}_t^j - M_{Bt}^j \right) \\ &= \sum_{j=0} \varphi_{t-j}^{\sigma-1} \bar{M}_t^j - y_{Bt}^{\sigma-1} \end{aligned}$$

so

$$y_{At}^{\sigma-1} + y_{Bt}^{\sigma-1} = \sum_{j=0} \varphi_{t-j}^{\sigma-1} \bar{M}_t^j.$$

This implies that, given a constant income y_{At}, any (small) reshuffling of capital goods that preserves market clearing conditions automatically delivers a constant income for country B too.

We complete the proof as follows. From a given an allocation of capital goods $\left\{ M_{At}^j, M_{Bt}^j \right\}_{j\geq 0}$, construct an alternative allocation of capital goods in country A, denoted $\left\{ \hat{M}_{At}^j \right\}_{j\geq 0}$, such that $y_{At} = \hat{y}_{At}$, and there are no large changes, i.e., $\left| M_{At}^j - \hat{M}_{At}^j \right| < \varepsilon$ for some $\varepsilon > 0$ and all $j \geq 0$. Such a change exists.

Set $\left\{ \hat{M}_{Bt}^j \right\}_{j\geq 0}$ such that the world market clearing conditions (13) are satisfied with unchanged production structure. Since the changes are small, we can use $M_{At}^j + M_{Bt}^j = \bar{M}^j$ for all j. The expression

$$y_{At}^{\sigma-1} + y_{Bt}^{\sigma-1} = \sum_{j=0} \varphi_{t-j}^{\sigma-1} \bar{M}_t^j.$$

implies that $\left\{ \hat{M}_{Bt}^j \right\}_{j\geq 0}$ delivers the same income in country B as $\left\{ M_{Bt}^j \right\}_{j\geq 0}$, $\hat{y}_{Bt} = y_{Bt}$.

4.4 An Example

We now provide a simple example to illustrate the equilibrium behavior of the distribution of capital vintages and the distribution of vintage prices along a balanced growth path. We will focus on an equilibrium with no trade in capital goods, so the composition of capital is indeed pinned down.

Recall that along a balanced growth path (BGP) the distribution of capital goods across vintages is constant, i.e., $\mu_t^j = \frac{M_t^j}{M_t} = \mu^j$. Hence, it follows that the stock of each vintage grows at the same rate as aggregate capital, i.e.,

$$\frac{M_{t+1}^j}{M_t^j} = \gamma_M. \tag{23}$$

As we have imposed the condition that capital goods prices must be constant along a balanced growth path (BGP), i.e., $f_t^j = f_{t+1}^j$, we must have $g_t^j\left(x_t^j\right) = g_{t+1}^j\left(x_{t+1}^j\right)$. Since $x_t^j = M_t^j$, this last condition implies that

$$g_t^j\left(\eta^j M_t\right) = g_{t+1}^j\left(\eta^j M_{t+1}\right)$$

where $\eta^j = \frac{N_t^j}{M_t}$. It is easy to check that under our conditions, η^j is constant along a BGP. Hence, if g_t^j is homogenous of some degree $\zeta > 0$ then we can always ensure the existence of a constant price path along any BGP. The homogeneity of g_t^j implies

$$g_t^j\left(\eta^j\right) = \gamma_M^\zeta g_{t+1}^j\left(\eta^j\right).$$

Note this implies that the cost of a fixed amount of capital goods is falling over time, $g_t^j\left(\eta^j\right) > g_{t+1}^j\left(\eta^j\right)$. Clearly, we can generalize our definition of a balanced growth path to accommodate any exogenous technological change on the production of capital goods.

The above properties map into an obvious choice for the g function:

$$g_t^j\left(x_t^j\right) = A_t^j\left(x_t^j\right)^\zeta$$

where $\zeta > 0$, $A_t^j > 0$. As discussed above, the constant prices implies a point condition on the growth rate of technology,

$$\frac{A_t^j}{A_{t-1}^j} = \gamma_M^{-\zeta}.$$

This is sufficient information to pin down the entire steady state distribution of prices and shares of capital vintages. We solve for the distribution of vintages among total and new machines for a given sequence of $\left\{A_t^j\right\}$. The following is the baseline choice of parameters. We work with a process for A_t^j of the form

$$A_t^j = \gamma_A A_t^{j-1}$$

with $\gamma_A = 1.02$. Hence, on any given date and for given levels of demand, each vintage is 2 percent cheaper than its previous version. For the shape of the distribution we do not really need to specify any level A_t^0. The remaining parameters are as follows: $\delta = .05$, $\gamma = 1.02$, $\sigma = 2.6$, $\zeta = .5$.

Figures 1-3 plot the frequency distributions of capital good prices, new capital goods and existing capital goods along the BGP. Figure 1 demonstrates that a capital good becomes cheaper as it becomes older. Figure 2 shows that the newer the vintage of a capital good the greater is its share in new investment. Lastly, Figure 3 shows the overall distribution of all different vintages along a BGP. The hump-shaped distribution of all capital goods is due to the fact that entry is occurring in not just the latest vintage but also older vintages. For our chosen parameterization, 10-year old machines have the highest share of total machines in the stationary steady state distribution.

We also plot the frequency distributions for two countries with different technologies. The blue country has $\gamma_A = 1.02$, the red country $\gamma_A = 1$ (constant tech). All remaining parameters are equal to the baseline choices for both countries. Figure 4 shows that amongst new capital goods bought at any date along the BGP, the blue country (which has positive technology growth in capital goods) has a higher share of newer vintages younger than age 11 than the red country and a smaller share of vintages older than 12 years. Correspondingly, Figure 5 shows that amongst all capital goods (old and new) in existence at any date along a BGP, the blue country has a larger share of capital goods younger than 20 years. Clearly, the blue country, in which newer vintages are cheaper to produce than older vintages on every date, ends up with a larger share of newer, more productive capital goods. Hence, its average productivity has to be higher as well relative to the red country

5 A Quantitative Evaluation

We now turn to evaluating the quantitative fit of the model relative to the data. Of particular interest to us is the implied income distribution of the model.

5.1 Using Price Data

The model allows us to generate income differences from readily available data on consumption prices. Recall that the income difference between any two countries is given by

$$\frac{y_t}{y_t'} = \left(\frac{f_t^{0\prime}}{f_t^0}\right)^{\frac{1}{\sigma-2}}.$$

The frontier machine, like all the other capital goods, is tradeable. We use the law of one price

$$f_t^0 = \varepsilon_t \phi_t^0$$

to equate the ratio of the cost of a frontier machine to the real exchange rate

$$\frac{f^0}{f^{0\prime}} = \frac{p_c'}{p_c}.$$

In other words, the nominal price (say in dollars) of a frontier machine is roughly constant across countries. Note that the real exchange rate in our model is just the cost of the consumption basket in the home country in terms of the cost of the same consumption basket in the numeraire country. Hence, the ratio of real exchange rates between any two countries is just the ratio of the cost of consumption in the two countries, i.e., the ratio of their p_c's.

The final step is to calibrate the elasticity of substitution σ. This is the key parameter for our cross-country results: the other parameters have no impact on income dispersion as long as they are constant across countries. For our baseline quantification of the model we set $\sigma = 2.6$ which is the value for the elasticity of demand for intermediate goods used by Acemoglu and Ventura (2003). We should note that since the capital income share in this model is σ^{-1}, setting $\sigma = 2.6$ implies a capital share of 0.38 which is close to the numbers reported by Gollin (2002).[10]

5.2 Predicted income distribution

We take our data from the Penn World table 6.2. There are 163 countries in the sample. We report results for three years – 1996, 2000, and 2002. We measure income differences by using data on output per worker. Every country's income is expressed relative to the United States. The resulting estimates for income dispersion are reported in Table 1.

[10] Our model implies that the cross-country relative income ratio is given by $w/w' = (f^d/f^{d'})^{\frac{1}{\sigma-2}}$. Using this relationship, we also ran a simple linear regression

$$\log\left(\frac{y_{it}}{y_{jt}}\right) = b \log\left(\frac{f_i^d}{f_j^d}\right) + \varepsilon$$

and then use $b = \frac{1}{2-\sigma}$. The estimate is around $\sigma = 2.5$ which is very close to our baseline parameterization.

Table 1. **GDP per worker,** $\sigma = 2.6$

Data: **Penn World Table 6.2**

	Std Dev		**Max/Min**	
	Data	Model	Data	Model
1996				
5 % censored	0.27	0.40	44	33
10 % censored	0.26	0.33	36	22
2000				
5 % censored	0.26	0.26	45	39
10 % censored	0.25	0.23	35	19
2002				
5 % censored	0.27	0.27	49	49
10 % censored	0.25	0.24	36	28

The numbers reported in the rows labelled "5% censored" show the results when we eliminate the richest 2.5% and the poorest 2.5% countries in our sample. The "10% censored" row has a corresponding interpretation. The table reports two sets of statistics – the standard deviation of relative incomes and the ratio of income of between the richest and poorest country in the sample. Thus, in 2000 for the 5% censored sub-sample, the standard deviation of relative incomes was 0.26 while ratio of incomes of the richest to the poorest countries was 44. The corresponding numbers generated by the model were 0.26 and 39. For the 10% censored sub-sample the standard deviation and richest to poorest income ratios in the data were 0.25 and 35 while the corresponding numbers generated by the model were 0.23 and 19. The numbers for 1996 and 2002 can be read off similarly. As the table makes clear, for all three years the standard deviation of relative incomes generated by the model is very close to the data. In terms of the ratio of incomes of the richest to the poorest countries in the relevant sample, while there is some variation across the different years, for all three years the model generates over 33-fold income differences between the richest and the poorest countries for the 5% censored sub-sample. We view these results as being surprisingly strong and broadly supportive of the model.

As was pointed out above, the key parameter for our model is the elasticity of substitution between intermediate goods, σ. As a robustness check we recompute our baseline results for GDP per worker for two different values: $\sigma = 2.5$, and 3. Table 2 reports the results for $\sigma = 2.5$ while Table 3 gives the results for $\sigma = 3$. Table 2 and 3 show two basic features. First, the ability of the model to reproduce the cross-country income dispersion is relatively robust to alternative values of σ. Even with $\sigma = 3$, the model generates a standard deviation of income which is almost the same as in the data. Contrarily, the fit of the model with respect to the income ratio of the richest to the poorest country in the sample declines as one increases the value of σ. Thus, for the 5 % percent censored sample in 2000, with $\sigma = 3$ the predicted max/min ratio of relative incomes from the model is 9 whereas in the data it is 45.[11]

We view the sensitivity of the relative income gap predictions with mixed feelings. Clearly, the fact that the relative income numbers move a lot with changes in σ suggest that it would be hard to identify exactly how much of the observed income gap the model is actually generating. That is a negative. On the positive side however, there are two ways to view this "excess" sensitivity result. First, note that $\sigma = 2.5$ implies a capital income share of 0.4 while $\sigma = 3$ implies a capital income share of 1/3. In the standard neoclassical model a capital income income share of 1/3 and $\frac{(K/Y)_{rich}}{(K/Y)_{poor}} = 3.6$, generates an income gap of only 1.9 while a capital share of 0.4 does only marginally better with an implied income gap of 2.4. Hence, in this range for the capital share, the standard model generates very small income gaps. In contrast, our model generates an income

[11] This is easy to see from equation (22) which says that $\frac{w}{w'} = \left(\frac{f^{d\prime}}{f^d}\right)^{\frac{1}{\sigma-2}}$. Hence, for $\sigma = 2.5$, the estimated relative price of frontier machines across countries is being raised to the power 2 whereas for $\sigma = 3$ the same relative price is only being raised to the power 1. Thus, the predicted income ratio under $\sigma = 3$ is only going to be the square root of the corresponding ratio under $\sigma = 2.5$.

gap of 9 even with $\sigma = 3$ (recall that $\sigma = 3$ implies an capital share of 1/3). This is over four times as large as the standard model. We see this as an improvement.

Second, our model takes an extreme stance in that all differences across countries are assumed to be captured through differences in relative investment goods prices. This is clearly an oversimplification since we are not accounting for factors such as human capital, institutions, preferences etc., etc.. In as much as these factors are important in accounting for cross-country differences, our quantitative results leave room for these explanations as well.

TABLE 2. **GDP per worker, ,** $\sigma = 2.5$

Data: **Penn World Table 6.2**

	Std Dev		**Max/Min**	
	Data	Model	Data	Model
1996				
5 % censored	0.27	0.44	44	67
10 % censored	0.26	0.36	36	41
2000				
5 % censored	0.26	0.26	45	81
10 % censored	0.25	0.22	35	34
2002				
5 % censored	0.27	0.28	49	108
10 % censored	0.25	0.23	36	55

TABLE 3. **GDP per worker, , $\sigma = 3$**

Data: **Penn World Table 6.2**

	Std Dev		**Max/Min**	
	Data	Model	Data	Model
1996				
5 % censored	0.27	0.30	44	8
10 % censored	0.26	0.26	36	6
2000				
5 % censored	0.26	0.26	45	9
10 % censored	0.25	0.25	35	6
2002				
5 % censored	0.27	0.27	49	10
10 % censored	0.25	0.25	36	7

We also study the fit of the induced world income distribution from the model. We plot the relative income per person in 2000 against the predicted series from the model with $\sigma = 2.6$. Figure 6 shows the fit: the scatter points are pretty tightly concentrated around the 45-degree line. The correlation between the predicted and the data series is 0.71. There is a large outlier in Japan, whose consumption price level is reported in the PWT as 50% higher than any other country. Not surprisingly, the model also underpredicts the income for many major oil-producers.[12] Overall, the correlation between the actual data and the model is above 70 percent for all years (1996, 2000, 2002) and most of the range considered for parameter σ. We conclude that the model fits the data quite well.

[12] These are quite easy to spot in Figure 6: Saudi Arabia (SAU), Brunei (BRN), Oman (OMN), Kuwait (KWT), and Qatar (QAT).

5.3 Predicted capital-output ratios

As we showed above, the model decomposes per capita income into two components – average productivity, $\tilde{\varphi}$, and machines, M. A number of studies report numbers for measured capital-output ratios. To map these reported capital stock numbers into our model we need a corresponding measure of capital in the model. Clearly, the number of machines is not an appropriate measure of the capital stock since different types of machines have different productivities and, hence, different prices associated with them.

One candidate measure which accounts for the quality differences between machines and weights them accordingly is

$$k = \sum_{j=0}^{\infty} \frac{f^j}{f^0} M^j$$

Note that dividing by f^0 converts $f^j M^j$ into international prices since f^j gives the price of machine j in terms of the domestic consumption basket of the country in question. This measure is aggregating the cost of each type of machine while using the price of that machine. The proposed measure can be rewritten as

$$k = M \sum_{j=0}^{\infty} \frac{f^j}{f^0} \frac{M^j}{M} = M \frac{\tilde{f}}{f^0}$$

where

$$\tilde{f} = \sum_{j=0}^{\infty} \mu^j f^j.$$

Recall that $\mu^j = \frac{M^j}{M}$ are the weights of the vintage distribution. Hence, one can write the ratio of capital-output ratios between two countries as

$$\frac{k/y}{k'/y'} = \frac{\tilde{f}M}{y} \frac{y'}{\tilde{f}'M'} \frac{f^{0\prime}}{f^0}.$$

But from equation (18) we know that $\frac{\tilde{f}M}{y} = A$ which is a constant. Hence,

$$\frac{k/y}{k'/y'} = \frac{f^{0\prime}}{f^0}.$$

A key feature of this expression is that it is independent of the precise pattern of capital goods trade across countries. This is important since we established above that the pattern of trade is indeterminate in the model.

Fig 7 plots the cross-country distribution of the implied capital-output ratios from the model against the corresponding numbers reported in Hall and Jones (1999). The correlation between the two series is 0.62. We should note that the model implies that productivity/quality differences in capital goods are captured perfectly by their prices. In the data this is unlikely to be so. This would account for some of the differences between our numbers and the data. Overall, we interpret these results as being supportive of the model.

5.4 Productivity Decomposition

A key motivation for this paper was to explain the large cross-country differences in total factor productivity (TFP) that have been found in many studies. Having formalized the model, we would now like to examine the model's implications for cross-country TFP patterns.

Assume that the data is being generated by the model that we have formalized here. In other words, suppose that the capital stock and output numbers that are reported in the data are actually being generated by our model. Suppose a researcher attempts to measure productivity across countries in this world by using a Cobb-Douglas production function:

$$y = A^{\frac{1}{1-\alpha}} \left(\frac{k}{y}\right)^{\frac{\alpha}{1-\alpha}} \tag{24}$$

where $y = Y/L$ is per capita output while α is the capital share and A denotes TFP. What would be the cross-country productivity gaps that this researcher would measure in the data? In the previous subsection we computed the k/y ratios implied by the model. Plugging those computed numbers along with the corresponding per capita output numbers computed by the model (see subsection 5.2 above) into the expression for per capita output in equation (24) gives the implied numbers for A. In computing these numbers we shall make the standard assumption that $\alpha = 1/3$.

Figure 8 plots the implied cross-country productivity differences that would be measured by the researcher using the Cobb-Douglas production function against the corresponding numbers reported by Hall and Jones (1999) who used this production function to measure productivity. The main difference between Hall-Jones and us is that we are using per capita output and k/y numbers that are generated by our model while they took these numbers from the data directly. The Figure shows that the implied productivity differences across countries that would be measured in the data even when the true data generating mechanism is our model are reasonably close to the numbers that are reported in studies which use the actual data, with the correlation between the two series being 0.73. We view this as suggestive of the fact that our model is generating output, capital and productivity distributions that are close to the numbers reported in the data.

6 Conclusion

In this paper we have formalized a model of embodied technology adoption which allows us to endogeneize total factor productivity (TFP). The main advantage of this approach is that it is able to generate larger cross-country income differences for the same given level of investment distortions. The primary mechanism is simple. A higher relative price of new capital goods reduces purchases of new capital goods. This margin is the same as in the standard disembodied technology model. The larger effect on income differences comes from the fact that a smaller share of new capital goods also implies a lower quality of the average capital in the economy. This reduces average productivity and hence, per capita income. Intuitively, the mechanism of the model reduces per capita income both along the intensive margin (the number of capital goods) as well as the quality margin (the average productivity of installed capital).

Based on price data from the PWT, we find that the predicted relative income series from the model fits the data quite well. The model replicates both the cross-country variation in relative incomes as well as the income disparity between the richest and the poorest countries of our sample. Moreover, the model generates a cross-country distribution of capital-output ratios that matches the data quite well. Lastly, we also found that the productivity dispersion that is generated by applying the lens of a Cobb-Douglas production function to data generated by our model matches the numbers reported in the data. We consider these quantitative results to be a qualified endorsement of the model.

In closing two comments are in order. First, we have taken an extreme position regarding the sources of productivity and income differences across countries; we have linked them exclusively to differences in physical capital stocks across countries. This clearly is too strong a position since one can easily imagine compelling reasons why differences in human capital or institutional quality may be important for cross-country productivity and income differences. From a theoretical perspective, it is straightforward to expand our formalization of capital or machines to also incorporate human capital. The data implementation of this augmented structure would be more complicated since one would now require a different measure of investment goods prices which also incorporates the cost of acquiring human capital. However, in as much as differences in the relative price of investment goods across countries also reflect the cross-country variation in institutional quality and/or the stocks of human capital (so that better institutions and higher stocks of human capital reduce the cost of investment), our results do capture these elements as well. Second, we have been silent on the reasons behind differences in investment prices across countries. There may be multiple reasons for these differences ranging from technology to policy-induced distortions. This is an important issue which we hope to address in future work.

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Figure 1: **Distribution of Capital Good Prices**

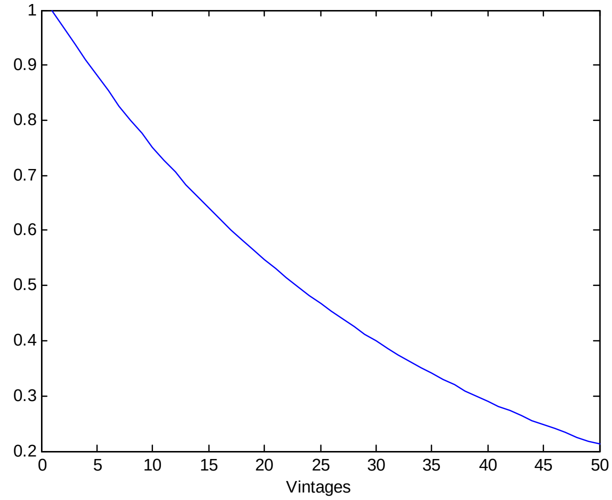

Figure 2: **Distribution of New Capital Goods along a BGP**

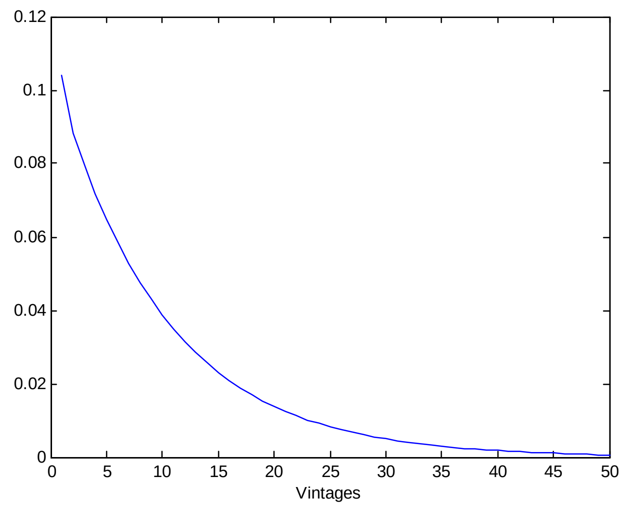

Figure 3: **Distribution of Existing Capital Goods**

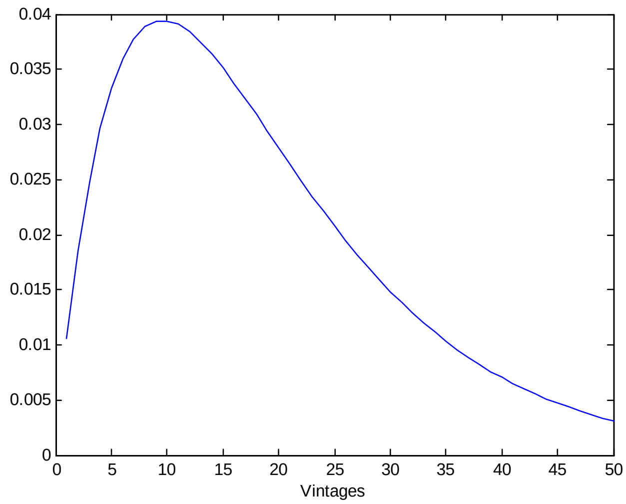

Figure 4: **Distribution of New Capital Goods** Blue line $\gamma_A = 1.02$, red line $\gamma_B = 1$.

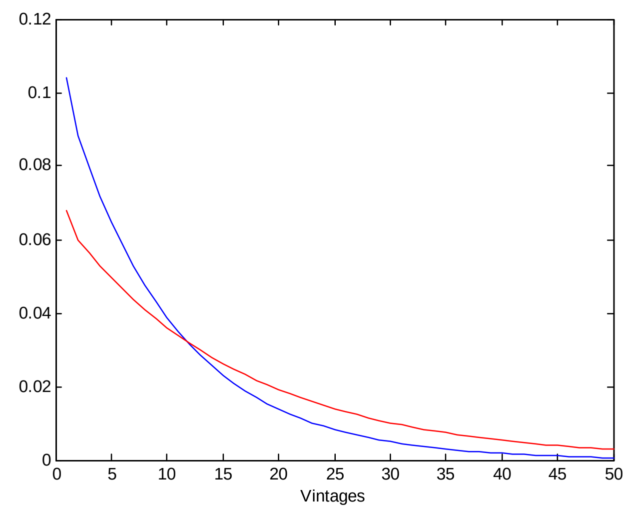

Figure 5: **Distribution of Existing Capital Goods**

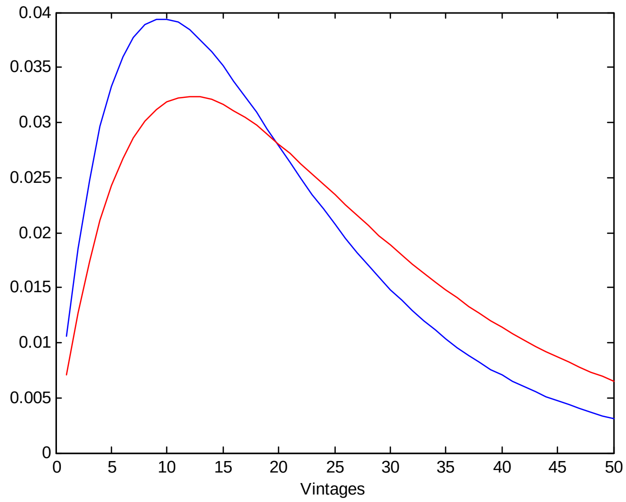

Figure 6: **Predicted Relative Incomes: Model and Data**

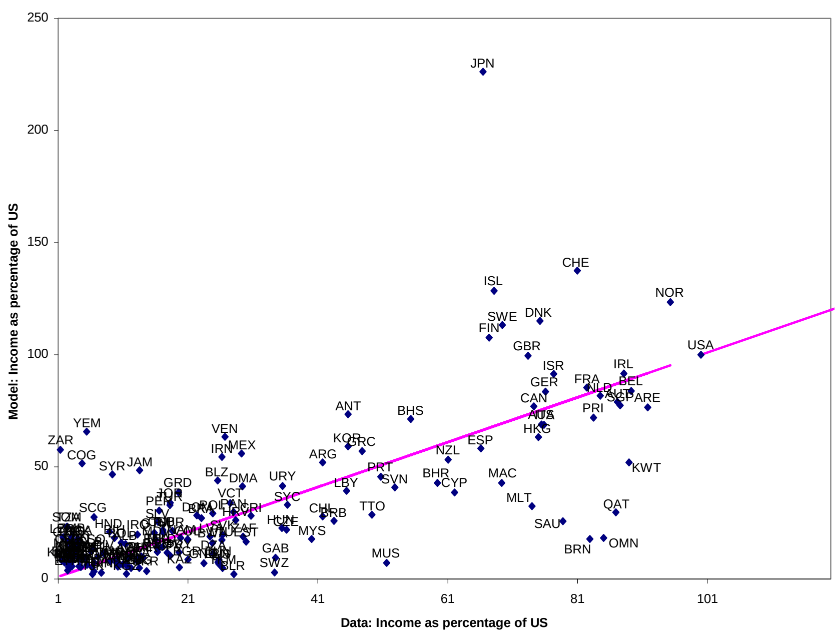

Figure 7: **Capital Output Ratios: Model and Data**

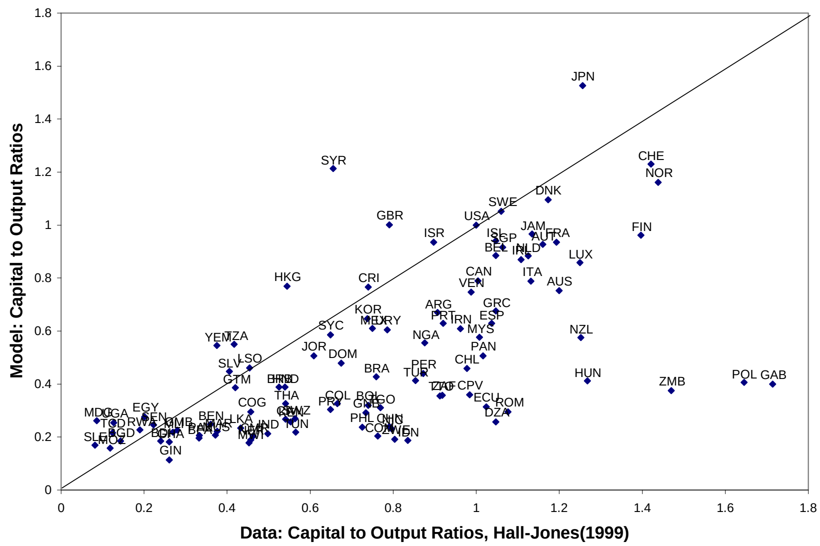

Figure 8: **Productivity Comparison: Model and Data**